(1) a) $\hat{H}' = -V_0$ for $x = 0 \rightarrow L/2$.

$\therefore E_1^{(1)} = \langle \psi_1 | \hat{H}' | \psi_1 \rangle$ and $\psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$

$$\therefore E_1^{(1)} = \frac{2}{L} \int_0^{L/2} \sin\left(\frac{\pi x}{L}\right)(-V_0) \sin\left(\frac{\pi x}{L}\right) dx = -\frac{2V_0}{L} \int_0^{L/2} \sin^2\left(\frac{\pi x}{L}\right) dx$$

note: integration is between 0 and $L/2$ $\because$ $\hat{H}' = \phi$ for $x > L/2$

From tables: $\int \sin^2(ax)\,dx = \frac{1}{2}x - \left(\frac{1}{4a}\right)\sin(2ax)$

$$\therefore E_1^{(1)} = -\frac{2V_0}{L}\left[\frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right]_0^{L/2}$$

$$= -\frac{2V_0}{L}\left[\frac{L}{4} - 0 - \left(\frac{L}{4\pi}\sin\left(\frac{2\pi L}{2L}\right) - \frac{L}{4\pi}\sin(0)\right)\right]$$

$$= -\frac{2V_0}{L}\left[\frac{L}{4} - \left(\frac{L}{4\pi}\underset{0}{\underset{\parallel}{\sin(\pi)}} - \frac{L}{4\pi}\underset{0}{\underset{\parallel}{\sin(0)}}\right)\right] = -\frac{2V_0L}{4L}$$

$$= \boxed{-\frac{V_0}{2} = E_1^{(1)}}$$

b) From spectral results $\psi_1^{(1)} = \sum_{k\neq 1} a_k \psi_k^{(0)}$

$$a_k = \frac{H_{k1}^{(1)}}{E_1^{(0)} - E_k^{(0)}}$$

First evaluate $H_{K1}^{(1)} = -\frac{2V_0}{L}\int_0^{L/2} \sin\left(\frac{K\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx \qquad (K \neq 1)$

From tables: $\int \sin(mx)\sin(nx)\,dx = \frac{\sin(m-n)x}{2(m-n)} - \frac{\sin(m+n)x}{2(m+n)}$

$$\Rightarrow -\frac{2V_0}{L}\int_0^{L/2} \sin\left(\frac{K\pi x}{L}\right)\sin\left(\frac{\pi x}{L}\right)dx = -\frac{2V_0}{\cancel{L}}\left[\frac{\sin\left[\frac{(K-1)\pi x}{L}\right]}{\frac{2\pi}{\cancel{L}}(K-1)} - \frac{\sin\left[\frac{(K+1)\pi x}{L}\right]}{\frac{2\pi}{L}(K+1)}\right]_0^{L/2}$$

$$= -\frac{V_0}{\pi}\left[\frac{\sin\left[\frac{(K-1)\pi}{2}\right] - \phi}{K-1} - \left\{\frac{\sin\left[\frac{(K+1)\pi}{2}\right] - \phi}{K+1}\right\}\right]$$

$$= -\frac{V_0}{\pi}\left[\frac{\sin\left[\frac{(K-1)\pi}{2}\right]}{K-1} - \frac{\sin\left[\frac{(K+1)\pi}{2}\right]}{K+1}\right]$$

Looks bad? Not really. Evaluate $H_{K1}^{(1)}$ for $K = 2, 3, 4, 5$.

$K=2 \qquad H_{2,1}^{(1)} = -\frac{V_0}{\pi}\left[\sin\left(\frac{\pi}{2}\right) - \frac{\sin\left(\frac{3\pi}{2}\right)}{3}\right] = -\frac{V_0}{\pi}\left[1 - \left(-\frac{1}{3}\right)\right] = -\frac{4V_0}{3\pi}$

$K=3 \qquad H_{3,1}^{(1)} = -\frac{V_0}{\pi}\left[\frac{\overbrace{\sin(\pi)}^{=\phi}}{2} - \frac{\overbrace{\sin(2\pi)}^{=\phi}}{4}\right] = \phi$

$K=4 \qquad H_{4,1}^{(1)} = -\frac{V_0}{\pi}\left[\frac{\sin\left(\frac{3\pi}{2}\right)}{3} - \frac{\sin\left(\frac{5\pi}{2}\right)}{5}\right] = -\frac{V_0}{\pi}\left[-\frac{1}{3} - \frac{1}{5}\right] = +\frac{8V_0}{15\pi}$

$K=5 \qquad H_{5,1}^{(1)} = -\frac{V_0}{\pi}\left[\frac{\overbrace{\sin(2\pi)}^{=\phi}}{4} - \frac{\overbrace{\sin(3\pi)}^{=\phi}}{6}\right] = \phi$

$\therefore$ only 2 terms are necessary.

$$\therefore \psi_1^{(1)} = a_2 \psi_2^{(0)} + a_4 \psi_4^{(0)}$$

$$a_2 = -\frac{4V_0}{3\pi}\left[\frac{1}{E_1^{(0)} - E_2^{(0)}}\right]. \quad \text{but } E_n^{(0)} = \frac{n^2 h^2}{8mL^2} \Rightarrow E_n^{(0)} = n^2 E_1^{(0)}$$

$$\therefore a_2 = -\frac{4V_0}{3\pi}\left[\frac{8mL^2}{(1-2^2)h^2}\right] = -\frac{4 \times 8 V_0 L^2 m}{3 \times (-3)\pi h^2} = +\frac{32 V_0 L^2 m}{9\pi h^2}$$

and $$a_4 = +\frac{8V_0}{15\pi}\left[\frac{1}{E_1^{(0)} - E_4^{(0)}}\right] = \frac{8V_0}{15\pi}\left[\frac{8mL^2}{(1-4^2)h^2}\right] = -\frac{64 V_0 L^2 m}{225\pi h^2}$$

$$\therefore \psi_1^{(1)} = \sqrt{\frac{2}{L}}\left[\frac{32 V_0 L^2 m}{9\pi h^2} \sin\left(\frac{2\pi x}{L}\right) - \frac{64 V_0 L^2 m}{225\pi h^2} \sin\left(\frac{4\pi x}{L}\right)\right]$$

$$\boxed{\psi_1^{(1)} = \frac{\sqrt{2}\, L^{1.5} V_0 m}{\pi h^2}\left[\frac{32}{9}\sin\left(\frac{2\pi x}{L}\right) - \frac{64}{225}\sin\left(\frac{4\pi x}{L}\right)\right]}$$

note : complete wavef^n is $\psi_1^{(0)} + \psi_1^{(1)}$

$$= \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) + \frac{\sqrt{2}\, L^{1.5} V_0 m}{\pi h^2}\left[\frac{32}{9}\sin\left(\frac{2\pi x}{L}\right) - \frac{64}{225}\sin\left(\frac{4\pi x}{L}\right)\right].$$