Solutions . Problem Set #4 2007

① a) Levels 1 - 3 are degenerate and ∴ secular determinant to solve is :

$$\begin{vmatrix} H_{11}^{(1)}-E^{(1)} & H_{12}^{(1)} & H_{13}^{(1)} \\ H_{21}^{(1)} & H_{22}^{(1)}-E^{(1)} & H_{23}^{(1)} \\ H_{31}^{(1)} & H_{23}^{(1)} & H_{33}^{(1)}-E^{(1)} \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} -E^{(1)} & 1 & 0 \\ 1 & -E^{(1)} & 1 \\ 0 & 1 & -E^{(1)} \end{vmatrix} = 0$$

Expand.

$$-E^{(1)}\begin{vmatrix} -E^{(1)} & 1 \\ 1 & -E^{(1)} \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 0 & -E^{(1)} \end{vmatrix} = 0$$

$$\Rightarrow -E^{(1)}\left(E^{(1)2}-1\right) - \left(-E^{(1)}\right) = 0 \Rightarrow -E^{(1)}\left(E^{(1)2}-2\right) = 0$$

$$\therefore \boxed{E^{(1)}=0} \quad , \quad E^{(1)2} = 2 \Rightarrow \boxed{E^{(1)} = \pm\sqrt{2}}$$

Find "correct" wavef^{n}s under perturbation

$\underline{E^{(1)}=0}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \therefore \quad c_2 = 0 \; , \; c_3 + c_1 = 0 \Rightarrow c_3 = -c_1$$

$$\therefore |\psi_1\rangle = c_1\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \; ; \text{ normalize} \Rightarrow \langle\psi_1|\psi_1\rangle = 1 = c_1^2(1 \;\; 0 \;\; -1)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$\therefore c_1^2(1+1) = 1 \Rightarrow 2c_1^2 = 1 \Rightarrow c_1 = 1/\sqrt{2} \Rightarrow |\psi_1\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$\underline{E^{(1)}=+\sqrt{2}}$

$$\begin{pmatrix} -\sqrt{2} & 1 & 0 \\ 1 & -\sqrt{2} & 0 \\ 0 & 1 & -\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \therefore \; -\sqrt{2}c_1 + c_2 = 0 \Rightarrow c_2 = \sqrt{2}c_1$$

$$c_2 - \sqrt{2}c_3 = 0 = c_3 = \frac{c_2}{\sqrt{2}} = \frac{\sqrt{2}c_1}{\sqrt{2}} = c_1$$

$$\therefore |\psi_2\rangle = c_1\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} \; ; \text{normalize } \langle\psi_2|\psi_2\rangle = c_1^2[1+2+1] = 1 = c_1^2 = \frac{1}{4}$$

$$c_1 = 1/2$$

$$\therefore \; |\psi_2\rangle = \begin{pmatrix} 1/2 \\ \sqrt{2}/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

$\underline{E^{(1)} = -\sqrt{2}}$

$$\begin{pmatrix} \sqrt{2} & 1 & 0 \\ 1 & \sqrt{2} & 0 \\ 0 & 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} c_2 = -\sqrt{2}\,c_1 \\ c_3 = -\dfrac{c_2}{\sqrt{2}} = -\dfrac{(-\sqrt{2})c_1}{\sqrt{2}} = c_1 \end{array}$$

$$\therefore |\psi_3\rangle = c_1 \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$; normalize $\langle\psi_3|\psi_3\rangle = 1 = c_1^2\,[1+2+1] = c_1^2 \cdot 4$

$\therefore c_1 = 1/2$

$$\therefore |\psi_3\rangle = \begin{pmatrix} 1/2 \\ -\sqrt{2}/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$$

Wavef^{ns} are normalized. Need to show they are orthogonal.

$$\langle\psi_1|\psi_2\rangle = \left(\tfrac{1}{\sqrt{2}} \;\; 0 \;\; -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0$$

$$\langle\psi_1|\psi_3\rangle = \left(\tfrac{1}{\sqrt{2}} \;\; 0 \;\; -\tfrac{1}{\sqrt{2}}\right)\begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = 0$$

$$\langle\psi_2|\psi_3\rangle = \left(\tfrac{1}{2} \;\; \tfrac{1}{\sqrt{2}} \;\; \tfrac{1}{2}\right)\begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix} = \frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0 .$$

b). $E_4^{(1)} = 2$ (read off $\hat{H}^{(1)}$ as $H_{44}^{(1)}$)

$$E_4^{(2)} = \sum_{k=1}^{3} \frac{|H_{k4}^{(1)}|^2}{E_4^{(0)} - E_k^{(0)}} = \frac{|\langle \frac{1}{\sqrt{2}}\psi_1 - \frac{1}{\sqrt{2}}\psi_3 | \hat{H}^{(1)} | \psi_4\rangle|^2}{2-1} + \frac{|\langle \frac{1}{2}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3 | \hat{H}^{(1)} | \psi_4\rangle|^2}{2-1}$$

$$+ \frac{|\langle \frac{1}{2}\psi_1 - \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3 \,|\, \hat{H}^{(1)} \,|\, \psi_4\rangle|^2}{2-1}$$

$$= \left|\frac{1}{\sqrt{2}}H_{14}^{(1)} - \frac{1}{\sqrt{2}}H_{34}^{(1)}\right|^2 + \left|\frac{1}{2}H_{14}^{(1)} + \frac{1}{\sqrt{2}}H_{24}^{(1)} + \frac{1}{2}H_{34}^{(1)}\right|^2 + \left|\frac{1}{2}H_{14}^{(1)} - \frac{1}{\sqrt{2}}H_{24}^{(1)} + \frac{1}{2}H_{34}^{(1)}\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\cdot 0 - \frac{1}{\sqrt{2}}\cdot 1\right|^2 + \left|\frac{1}{2}\cdot 0 + \frac{1}{\sqrt{2}}\cdot 0 + \frac{1}{2}\cdot 1\right|^2 + \left|\frac{1}{2}\cdot 0 - \frac{1}{\sqrt{2}}\cdot 0 + \frac{1}{2}\cdot 1\right|^2$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1$$

$$\therefore E_4 = E_4^{(0)} + E_4^{(1)} + E_4^{(2)} = 2 + 2 + 1 = \boxed{5}$$

2.

$$\hat{H}^{(1)} = -qEx \qquad 0 \le t \le \tau$$

$$\hat{H}^{(1)} = 0 \qquad \text{all times} > \tau$$

time dependent perturbation theory finds

$$a_q^{(1)}(t) = \frac{-i}{\hbar}\int_0^t H_{qj}^{(1)} e^{-i\omega_{jq}t}\,dt, \qquad \hbar\omega_{jq} = E_j^{(0)} - E_q^{(0)}$$

First evaluate $H_{qj}^{(1)}$ for the given problem. Ground state of oscillator

$$|j\rangle = |0\rangle$$

If $|q\rangle = |1\rangle \Rightarrow H_{10}^{(1)} = \langle 1|\hat{H}^{(1)}|0\rangle = -qE\langle 1|x|0\rangle$

Recall: $\hat{x} = \frac{1}{2}\sqrt{\frac{2}{\alpha}}(\hat{a}^+ + \hat{a})$; $\alpha = \frac{\omega_0 m}{\hbar}$

$$\therefore \hat{H}_{10}^{(1)} = -\frac{qE}{2}\sqrt{\frac{2\hbar}{m\omega_0}}\langle 1|\hat{a}^+ + \hat{a}|0\rangle = -\frac{qE}{2}\sqrt{\frac{2\hbar}{m\omega_0}}\left[\langle 1|\hat{a}^+|0\rangle + \langle 1|\hat{a}|0\rangle\right]$$

$$= -\frac{qE}{2}\sqrt{\frac{2\hbar}{m\omega_0}}[1+0] = -\frac{qE}{2}\sqrt{\frac{2\hbar}{m\omega_0}}$$

$$\therefore a_1^{(1)}(t) = \frac{iqE}{2\hbar}\sqrt{\frac{2\hbar}{m\omega_0}}\int_0^\tau e^{-i\omega_{01}t}\,dt$$

$$\omega_{01} = \frac{E_0^{(0)} - E_1^{(0)}}{\hbar} = \frac{\hbar\omega_0(\frac{1}{2}) - \hbar\omega_0(\frac{3}{2})}{\hbar} = -\omega_0$$

$$\therefore a_1^{(1)}(t) = \frac{iqE}{2}\sqrt{\frac{2}{m\omega_0\hbar}}\int_0^\tau e^{i\omega_0 t}\,dt = \frac{iqE}{2}\sqrt{\frac{2}{m\omega_0\hbar}}\left.\frac{e^{i\omega_0 t}}{i\omega_0}\right|_0^\tau$$

$$= \frac{iqE}{2i\omega_0}\sqrt{\frac{2}{m\omega_0\hbar}}\left(e^{i\omega_0\tau}-1\right) = qE\sqrt{\frac{1}{2m\omega_0^3\hbar}}\left(e^{i\omega_0\tau}-1\right)$$

Rewrite as

$$qE\sqrt{\frac{1}{2m\omega_0^3\hbar}}\left[e^{i\omega_0\tau/2}e^{i\omega_0\tau/2} - e^{-i\omega_0\tau/2}e^{i\omega_0\tau/2}\right]\times\frac{2i}{2i}$$

$$= qE\sqrt{\frac{1}{2m\omega_0^3\hbar}}\, e^{i\omega_0\tau/2}\cdot 2i\,\sin\left(\frac{\omega_0\tau}{2}\right) \qquad \because \frac{e^{ix}-e^{-ix}}{2i} = \sin x$$

$$\therefore P_{1\leftarrow 0} = |a_1^{(1)}(t)|^2 = \frac{q^2E^2\cdot 4}{2m\omega_0^3\hbar}\sin^2\left(\frac{\omega_0\tau}{2}\right) = \boxed{\frac{2q^2E^2}{m\omega_0^3\hbar}\sin^2\left(\frac{\omega_0\tau}{2}\right)}$$

b) $H_{20}^{(1)} \propto \langle 2|x|0\rangle = 0$ by properties of $\hat{a}, \hat{a}^\dagger$ at all times

$\Rightarrow a_2^{(1)}(t) = 0$ for all times $\Rightarrow P_{2\leftarrow 0} = 0$ at all times