Solutions : Problem Set #6 C474b 2007

1 a) First couple electrons one and two $\ell_1 = \ell_2 = 2$

$\therefore L' = |\ell_1 + \ell_2| - |\ell_1 - \ell_2| = |2+2| - |2-2| = 4, 3, 2, 1, 0$

next couple each L' with $\ell_3 = 2$

$\therefore L = |4+2| - |4-2| = 6, 5, 4, 3, 2$

$L = |3+2| - |3-2| = 5, 4, 3, 2, 1$

$L = |2+2| - |2-2| = 4, 3, 2, 1, 0$

$L = |1+2| - |1-2| = 3, 2, 1$

$L = |0+2| - |0-2| \quad 2$

Values of total angular momentum are

$L = 6(\times 1), 5(2\times), 4(3\times), 3(4\times), 2(5\times), 1(3\times), 0(\times 1)$

b) First couple electrons one and two $s_1 = s_2 = \frac{1}{2}$

$S' = |s_1 + s_2| - |s_1 - s_2| = |\frac{1}{2} + \frac{1}{2}| - |\frac{1}{2} - \frac{1}{2}| = 1, 0$

next couple each s' with $s_3 = \frac{1}{2}$

$\therefore S = |1 + \frac{1}{2}| - |1 - \frac{1}{2}| = 3/2, \frac{1}{2}$

$S = |0 + \frac{1}{2}| - |0 - \frac{1}{2}| = \frac{1}{2}$

Values of total spin are $S = 3/2$, $S = \frac{1}{2}\ (2\times)$.

c) Total angular momentum $= J = |L+S| - |L-S|$ for each L and S.

degeneracy $= (2L+1)(2S+1)$

$\therefore J = |6 + 3/2| - |6 - 3/2| = 15/2, 13/2, 11/2, 9/2 \quad (\times 1)$

$J = |6 + \frac{1}{2}| - |6 - \frac{1}{2}| = 13/2, 11/2 \quad (\times 2)$

$J = |5 + 3/2| - |5 - 3/2| = 13/2, 11/2, 9/2, 7/2 \quad (\times 2)$

$J = |5 + \frac{1}{2}| - |5 - \frac{1}{2}| = 11/2, 9/2 \quad (\times 4)$

$J = |4 + 3/2| - |4 - 3/2| = 11/2, 9/2, 7/2, 5/2 \quad (\times 3)$

$J = |4 + \frac{1}{2}| - |4 - \frac{1}{2}| = 9/2, 7/2 \quad (\times 6)$

$J = |3 + 3/2| - |3 - 3/2| = 9/2, 7/2, 5/2, 3/2 \quad (\times 4)$

$J = |3 + \frac{1}{2}| - |3 - \frac{1}{2}| = 7/2, 5/2 \quad (\times 8)$

$J = |2 + 3/2| - |2 - 3/2| = 7/2, 5/2, 3/2, \frac{1}{2} \quad (\times 5)$

$J = |2 + \frac{1}{2}| - |2 - \frac{1}{2}| = 5/2, 3/2 \quad (\times 10)$

$J = |1 + 3/2| - |1 - 3/2| = 5/2, 3/2, \frac{1}{2} \quad (\times 3)$

$J = |1 + \frac{1}{2}| - |1 - \frac{1}{2}| = 3/2, \frac{1}{2} \quad (\times 6)$

$J = |0 + 3/2| - |0 - 3/2| = 3/2 \quad (\times 1)$

$J = |0+\frac{1}{2}| - |0-\frac{1}{2}| = \frac{1}{2}$ (x2)

$\therefore J = 15/2$ (x1), $13/2$ (x5), $11/2$ (x12), $9/2$ (x20), $7/2$ (x28), $5/2$ (x33)

$3/2$ (x29), $1/2$ (x16)   163 of states.

d) $j_i = |l_i + s_i| - |l_i - s_i| = |2+\frac{1}{2}| - |2-\frac{1}{2}| = 5/2, 3/2$ for each electron

first couple j_1 and j_2 $\Rightarrow$ $J' = |5/2+5/2| - |5/2-5/2| = 5, 4, 3, 2, 1, 0$
$J' = |5/2+3/2| - |5/2-3/2| = 4, 3, 2, 1$
$J' = |3/2+5/2| - |3/2-5/2| = 4, 3, 2, 1$
$J' = |3/2+3/2| - |3/2-3/2| = 3, 2, 1, 0$

$\therefore J' = 5$ (x1), 4 (x3), 3 (x4), 2 (x4), 1 (x4), 0 (x2)

Now couple each j_3 with each J' to form J

$J = |5+5/2| - |5-5/2| = 15/2, 13/2, 11/2, 9/2, 7/2, 5/2$ (x1)
$J = |5+3/2| - |5-3/2| = 13/2, 11/2, 9/2, 7/2$ (x1)
$J = |4+5/2| - |4-5/2| = 13/2, 11/2, 9/2, 7/2, 5/2, 3/2$ (x3)
$J = |4+3/2| - |4-3/2| = 11/2, 9/2, 7/2, 5/2$ (x3)
$J = |3+5/2| - |3-5/2| = 11/2, 9/2, 7/2, 5/2, 3/2, 1/2$ (x4)
$J = |3+3/2| - |3-3/2| = 9/2, 7/2, 5/2, 3/2$ (x4)
$J = |2+5/2| - |2-5/2| = 9/2, 7/2, 5/2, 3/2, 1/2$ (x4)
$J = |2+3/2| - |2-3/2| = 7/2, 5/2, 3/2, 1/2$ (x4)
$J = |1+5/2| - |1-3/2| = 7/2, 5/2, 3/2$ (x4)
$J = |1+3/2| - |1-3/2| = 5/2, 3/2, 1/2$ (x4)
$J = |0+5/2| - |0-5/2| = 5/2$ (x2)
$J = |0+3/2| - |0-3/2| = 3/2$ (x2)

$\therefore J = 15/2$ (x1), $13/2$ (x5), $11/2$ (x12), $9/2$ (x20), $7/2$ (x28), $5/2$ (x33)

$3/2$ (x29), $1/2$ (x16)

$\therefore$ # states are the SAME as part c) as is the distribution.

② Let uncoupled wavefunctions be $|l_1, m_{l_1}, l_2, m_{l_2}\rangle = |m_{l_1}, m_{l_2}\rangle$

Let coupled wavefunctions be $|l_1, l_2, j, m_j\rangle = |j, m_j\rangle$

→ $|j, m_j\rangle = |2,2\rangle$ can be formed in only 1 way : $m_{l_1} = +l_1 = +1$; $m_{l_2} = +l_2 = +1$

⇒ $\boxed{|2,2\rangle = |1,1\rangle}$ ⇒ $C_{m_{l_1}, m_{l_2}} = \boxed{C_{1,1} = 1}$

→ Similarly $|j, m_j\rangle = |2,-2\rangle$ can be formed in only 1 way : $m_{l_1} = -l_1 = -1$; $m_{l_2} = -l_2 = -1$

⇒ $\boxed{|2,-2\rangle = |-1,-1\rangle}$ → $\boxed{C_{-1,-1} = 1}$

→ To get $|2,1\rangle$ use $\hat{J}_- = \hat{L}_{1-} + \hat{L}_{2-}$ on $|2,2\rangle = |1,1\rangle$

LHS: $\hat{J}_-|2,2\rangle = \sqrt{(2)(3)-(2)(1)}\,|2,1\rangle = 2\hbar|2,1\rangle$

RHS: $(\hat{L}_{1-} + \hat{L}_{2-})|1,1\rangle = \hat{L}_{1-}|1,1\rangle + \hat{L}_{2-}|1,1\rangle$

$= \sqrt{(1)(2)+(1)(1-1)}\,\hbar|0,1\rangle + \sqrt{(1)(2)+(1)(1-1)}\,\hbar|1,0\rangle = \sqrt{2}\hbar|0,1\rangle + \sqrt{2}\hbar|1,0\rangle$

⇒ $|2,1\rangle = \frac{\sqrt{2}}{2}|0,1\rangle + \frac{\sqrt{2}}{2}|1,0\rangle$ ⇒ $\boxed{|2,1\rangle = \frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle}$

⇒ $\boxed{C_{0,1} = 1/\sqrt{2}}$ and $\boxed{C_{1,0} = 1/\sqrt{2}}$

→ To get $|2,0\rangle$ use $\hat{J}_- = \hat{L}_{1-} + \hat{L}_{2-}$ on $|2,1\rangle = \frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle$

$\therefore \hat{J}_-|2,1\rangle = (\hat{L}_{1-} + \hat{L}_{2-})\left(\frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle\right) = \frac{1}{\sqrt{2}}\hat{L}_{1}|0,1\rangle + \frac{1}{\sqrt{2}}\hat{L}_{1-}|1,0\rangle + \frac{1}{\sqrt{2}}\hat{L}_{2-}|0,1\rangle + \frac{1}{\sqrt{2}}\hat{L}_{2-}|1,0\rangle$

LHS: $\hat{J}_-|2,1\rangle = \sqrt{(2)(3)+0(0-1)}\,|2,0\rangle = \sqrt{6}\hbar|2,0\rangle$

RHS: $\frac{1}{\sqrt{2}}\sqrt{(1)(2)-0(0-1)}\,\hbar|-1,1\rangle + \frac{1}{\sqrt{2}}\sqrt{(1)(2)-1(1-1)}\,\hbar|0,0\rangle + \frac{1}{\sqrt{2}}\sqrt{(1)(2)-(1)(1-1)}\,\hbar|0,0\rangle + \frac{1}{\sqrt{2}}\sqrt{(1)(2)-0(0-1)}\,\hbar|1,-1\rangle$

$= \frac{\sqrt{2}}{\sqrt{2}}\hbar\,|{-1},1\rangle + \frac{\sqrt{2}}{\sqrt{2}}\hbar\,|0,0\rangle + \frac{\sqrt{2}}{\sqrt{2}}\hbar\,|0,0\rangle + \frac{\sqrt{2}}{\sqrt{2}}\hbar\,|1,-1\rangle$

$= \hbar\,|{-1},1\rangle + \hbar\,|0,0\rangle + \hbar\,|0,0\rangle + \hbar\,|1,-1\rangle = \hbar\,|{-1},1\rangle + 2\hbar\,|0,0\rangle + \hbar\,|1,-1\rangle$

$\Rightarrow \boxed{|2,0\rangle = \frac{1}{\sqrt{6}}|{-1},1\rangle + \frac{2}{\sqrt{6}}|0,0\rangle + \frac{1}{\sqrt{6}}|1,-1\rangle} \Rightarrow \boxed{C_{-1,1} = 1/\sqrt{6}}$ and $\boxed{C_{0,0} = \frac{2}{\sqrt{6}}}$ and $\boxed{C_{1,-1} = 1/\sqrt{6}}$

To get $|2,-1\rangle$ either operate with $\hat{J}_- = \hat{L}_{1-} + \hat{L}_{2-}$ on $|2,0\rangle$ <u>or</u> operate on $|2,-2\rangle$ with $\hat{J}_+ = \hat{L}_{1+} + \hat{L}_{2+}$. The latter is easier, so I'll do that. Either way is OK. $\quad |2,-2\rangle = |{-1},-1\rangle$

LHS: $\therefore \hat{J}_+|2,-2\rangle = \sqrt{(2)(3) - (-2)(-2+1)}\,\hbar\,|2,-1\rangle = \sqrt{6-2}\,\hbar\,|2,-1\rangle = 2\hbar\,|2,-1\rangle$

RHS: $(\hat{L}_{1+} + \hat{L}_{2+})|{-1},-1\rangle = \hat{L}_{1+}|{-1},-1\rangle + \hat{L}_{2+}|{-1},-1\rangle = \sqrt{(1)(2) - (-1)(-1+1)}\,\hbar\,|0,-1\rangle + \sqrt{(1)(2)-(-1)(-1+1)}\,\hbar\,|{-1},0\rangle$

$= \sqrt{2}\,\hbar\,|0,-1\rangle + \sqrt{2}\,\hbar\,|{-1},0\rangle$

$\therefore |2,-1\rangle = \frac{\sqrt{2}}{2}|0,-1\rangle + \frac{\sqrt{2}}{2}|{-1},0\rangle \Rightarrow \boxed{|2,-1\rangle = \frac{1}{\sqrt{2}}|0,-1\rangle + \frac{1}{\sqrt{2}}|{-1},0\rangle}$

$\therefore \boxed{C_{0,-1} = 1/\sqrt{2}}$ and $\boxed{C_{-1,0} = 1/\sqrt{2}}$

<u>Next do J=1 manifold</u>

Let $|1,1\rangle = a_1|0,1\rangle + a_2|1,0\rangle$ and require $\langle 1,1|1,1\rangle = 1$ and $\langle 2,1|1,1\rangle = \phi$

$\langle 1,1|1,1\rangle \Rightarrow a_1^2 + a_2^2 = 1 \qquad \langle 2,1|1,1\rangle \Rightarrow \frac{a_1}{\sqrt{2}} + \frac{a_2}{\sqrt{2}} = \phi \Rightarrow a_2 = -a_1$

$\therefore 2a_1^2 = 1 \Rightarrow a_1 = 1/\sqrt{2} \quad a_2 = -1/\sqrt{2}$

$\therefore \boxed{|1,1\rangle = \frac{1}{\sqrt{2}}|0,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle} \Rightarrow \boxed{C_{0,1} = 1/\sqrt{2}}$ and $\boxed{C_{1,0} = \frac{-1}{\sqrt{2}}}$

$\rightarrow$ get $|1,0\rangle$ by applying $\hat{J}_- = \hat{L}_{1-} + \hat{L}_{2-}$ to $|1,1\rangle = 1/\sqrt{2}\,|0,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle$

$$\underline{\text{LHS}}: \quad \hat{J}_-|1,1\rangle = \sqrt{(1)(2)-(1)(1-1)}\,\hbar\,|1,0\rangle = \sqrt{2}\,\hbar\,|1,0\rangle$$

$$\underline{\text{RHS}}: \quad (\hat{L}_{1-}+\hat{L}_{2-})\left(\frac{1}{\sqrt{2}}|0,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle\right) = \frac{1}{\sqrt{2}}\hat{L}_{1-}|0,1\rangle - \frac{1}{\sqrt{2}}\hat{L}_{1-}|1,0\rangle + \frac{1}{\sqrt{2}}\hat{L}_{2-}|0,1\rangle - \frac{1}{\sqrt{2}}\hat{L}_{2-}|1,0\rangle$$

$$= \frac{1}{\sqrt{2}}\sqrt{(1)(2)-0(0-1)}\,\hbar\,|-1,1\rangle - \frac{1}{\sqrt{2}}\sqrt{(1)(2)-1(1-1)}\,\hbar\,|0,0\rangle + \frac{1}{\sqrt{2}}\sqrt{(1)(2)-1(1-1)}\,\hbar\,|0,0\rangle - \frac{1}{\sqrt{2}}\sqrt{(1)(2)-0(0-1)}\,\hbar\,|1,-1\rangle$$

$$= \frac{\sqrt{2}}{\sqrt{2}}\hbar|-1,1\rangle - \cancel{\frac{\sqrt{2}}{\sqrt{2}}\hbar|0,0\rangle} + \cancel{\frac{\sqrt{2}}{\sqrt{2}}\hbar|0,0\rangle} - \frac{\sqrt{2}}{\sqrt{2}}\hbar|1,-1\rangle = \hbar|-1,1\rangle - \hbar|1,-1\rangle$$

$$\Rightarrow \boxed{|1,0\rangle = \frac{1}{\sqrt{2}}|-1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle} \quad \therefore \boxed{C_{-1,1} = \frac{1}{\sqrt{2}}} \text{ and } \boxed{C_{1,-1} = -\frac{1}{\sqrt{2}}}$$

$\rightarrow$ To get $|1,-1\rangle$ apply $\hat{J}_- = \hat{L}_{1-} + \hat{L}_{2-}$ to $|1,0\rangle = 1/\sqrt{2}\,|-1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle$

$$\underline{\text{LHS}}: \quad \hat{J}_-|1,0\rangle = \sqrt{(1)(2)-\phi(0-1)}\,\hbar\,|1,-1\rangle = \sqrt{2}\,\hbar\,|1,-1\rangle$$

$$\underline{\text{RHS}}: \quad (\hat{L}_{1-}+\hat{L}_{2-})\left(\frac{1}{\sqrt{2}}|-1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle\right) = \frac{1}{\sqrt{2}}\hat{L}_{1-}|-1,1\rangle - \frac{1}{\sqrt{2}}\hat{L}_{1-}|1,-1\rangle + \frac{1}{\sqrt{2}}\hat{L}_{2-}|-1,1\rangle - \frac{1}{\sqrt{2}}\hat{L}_{2-}|1,-1\rangle$$

$$= \phi - \frac{1}{\sqrt{2}}\sqrt{(1)(2)-(1)(1-1)}\,\hbar\,|0,-1\rangle + \frac{1}{\sqrt{2}}\sqrt{(1)(2)-(1)(1-1)}\,\hbar\,|-1,0\rangle + \phi$$

$$= -\frac{\sqrt{2}}{\sqrt{2}}\hbar|0,-1\rangle + \frac{\sqrt{2}}{\sqrt{2}}\hbar|-1,0\rangle = \hbar|-1,0\rangle - \hbar|0,-1\rangle$$

$$\therefore \boxed{|1,-1\rangle = \frac{1}{\sqrt{2}}|-1,0\rangle - \frac{1}{\sqrt{2}}|0,-1\rangle} \quad \therefore \boxed{C_{-1,0} = \frac{1}{\sqrt{2}}} \text{ and } \boxed{C_{0,-1} = -\frac{1}{\sqrt{2}}}$$

Finally to get $|j, m_j\rangle = |0,0\rangle$

Let $|0,0\rangle = a_1|-1,1\rangle + a_2|0,0\rangle + a_3|1,-1\rangle$ and require

$\langle 0,0|0,0\rangle = 1$ and $\langle 2,0|0,0\rangle = \phi$ $\underline{\text{and}}$ $\langle 1,0|0,0\rangle = \phi$

$$\langle 0,0|0,0\rangle = 1 \Rightarrow a_1^2 + a_2^2 + a_3^2 = 1$$

$$\langle 2,0|0,0\rangle = \phi \Rightarrow \frac{a_1}{\sqrt{6}} + \frac{2a_2}{\sqrt{6}} + \frac{a_3}{\sqrt{6}} = \phi$$

$\langle 1,0|0,0\rangle = \phi = \frac{a_1}{\sqrt{2}} + \phi a_2 - \frac{a_3}{\sqrt{2}} = \phi \Rightarrow a_3 = a_1$

$\therefore \frac{a_1}{\sqrt{6}} + \frac{2a_2}{\sqrt{6}} + \frac{a_1}{\sqrt{6}} = \phi \Rightarrow \frac{2a_1}{\sqrt{6}} + \frac{2a_2}{\sqrt{6}} = \phi \Rightarrow a_2 = -a_1$

$\therefore a_1^2 + a_1^2 + a_1^2 = 1 \Rightarrow 3a_1^2 = 1 \Rightarrow a_1 = 1/\sqrt{3} \therefore a_2 = -\frac{1}{\sqrt{3}} \quad a_3 = \frac{1}{\sqrt{3}}$

$\therefore \boxed{|0,0\rangle = \frac{1}{\sqrt{3}}|-1,1\rangle - \frac{1}{\sqrt{3}}|0,0\rangle + \frac{1}{\sqrt{3}}|1,-1\rangle}$

$\therefore \boxed{C_{-1,1} = 1/\sqrt{3}}$ and $\boxed{C_{0,0} = -\frac{1}{\sqrt{3}}}$ and $\boxed{C_{1,-1} = \frac{1}{\sqrt{3}}}$

(3) a) Clebsch – Gordon series $J = (\ell+s) - |\ell-s|$

$\ell = 2$ (D state) $s = 1/2$

$J = (2+1/2) - |2-1/2| = 5/2, 3/2$

b) $E_J = \langle \ell, s, j, m_j | \hat{H} | \ell, s, j, m_j\rangle$ in the coupled representation

$\therefore E_J = \langle \ell, s, j, m_j | A + \frac{B}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2) + C\hat{L}^2 | \ell, s, j, m_j\rangle$

$\because$ from class $\hat{L}\cdot\hat{S} = \frac{1}{2}(\hat{J}^2 - \hat{L}^2 - \hat{S}^2)$

$= A\langle \ell,s,j,m_j|\ell,s,j,m_j\rangle + \frac{B}{2}\langle \ell,s,j,m_j|\hat{J}^2 - \hat{S}^2|\ell,s,j,m_j\rangle$

$+ (C - \frac{B}{2})\langle \ell,s,j,m_j|\hat{L}^2|\ell,s,j,m_j\rangle$

$= A + \frac{B}{2}[j(j+1) - s(s+1)]\hbar^2 + (C - \frac{B}{2})\ell(\ell+1)\hbar^2$

$\Rightarrow \ell = 2, \; s = 1/2, \; j = 5/2$

$E_{5/2} = A + \frac{B}{2}[(5/2)(7/2) - (1/2)(3/2)]\hbar^2 + (C - \frac{B}{2})(2)(3)\hbar^2$

$= \boxed{A + B\hbar^2 + 6C\hbar^2}$

$\Rightarrow \ell = 2, \; s = 1/2 \; j = 3/2$

$E_{3/2} = A + \frac{B}{2}[(3/2)(5/2) - (1/2)(3/2) - (3)(2)]\hbar^2 + C(2)(3)\hbar^2$

$= \boxed{A - \frac{3B\hbar^2}{2} + 6C\hbar^2}$