5.2 Coupling of more than 2 angular momenta: j_1, j_2, j_3,...

1.) Couple first two angular momenta $\mathbf{j}_1$ and $\mathbf{j}_2$ as before to generate $\mathbf{J}$'

2.) Then couple these results individually to $\mathbf{j}_3$ to get set of $\mathbf{J}$'s

Example: $\ell_1 = 1;\ \ell_2 = 1;\ \ell_3 = 2$

Couple ℓ_1 and ℓ_2 to give L_{12}

$$\vec{\ell}_1 + \vec{\ell}_2 = (1+1) \rightarrow |1-1| = 2 \rightarrow 0 = 2,1,0 = \vec{L}_{12}$$

$$\begin{aligned}\vec{L}_{12} + \vec{\ell}_3 = \vec{L} &= (2+2) \rightarrow |2-2| = 4 \rightarrow 0 = 4,3,2,1,0 \\ &= (1+2) \rightarrow |1-2| = 3 \rightarrow 1 = 3,2,1 \\ &= (0+2) \rightarrow |0-2| = 2\end{aligned}$$

$$\therefore L = 4,3(\times 2),2(\times 3),1(\times 2),0$$

9 different angular momentum states, although some have the same total L values.

It should be and in fact it is possible to represent the coupled wave functions in terms of the uncoupled ones:

$$| j_1, j_2, J, M_J >= \sum_{m_{j1}} \sum_{m_{j2}} C_{m_{j1}, m_{j2}} | j_1, m_{j_1}, j_2, m_{j_2} >$$

The coefficients in this expansion are called **Clebsch-Gordon coefficients** or Wigner coefficients

Note: $\left\langle j_1', m_{j_1}'; j_2', m_{j_2}' \middle| j_1, j_2, J, M_J \right\rangle = \left\langle j_1', m_{j_1}'; j_2', m_{j_2}' | \sum_{m_{j1}} \sum_{m_{j2}} C_{m_{j_1}, m_{j_2}} \middle| j_1, m_{j_1}; j_2, m_{j_2} \right\rangle$

$$= \sum_{m_{j1}} \sum_{m_{j2}} C_{m_{j_1}, m_{j_2}} \left\langle j_1', m_{j_1}'; j_2', m_{j_2}' \middle| j_1, m_{j_1}; j_2, m_{j_2} \right\rangle$$

$$= C_{m_{j_1}', m_{j_2}'}$$

Represents the degree of coupling or overlap

Arrangement of product functions according to M values (eigenvalues of $L_{z,\,tot}$)

$M=l_1+l_2$	$\|l_1,1_1>\|l_{2,},l_2>$			…1
$M=l_1+l_2-1$	$\|l_1,1_1-1>\|l_{2,},l_2>$	$\|l_1,1_1>\|l_{2,},l_2-1>$		…2
$M=l_1+l_2-2$	$\|l_1,1_1-2>\|l_{2,},l_2>$	$\|l_1,1_1-1>\|l_{2,},l_2-1>$	$\|l_1,1_1>\|l_{2,},l_2-2>$	…3
…….	……..	……..	……..	
$M=-l_1-l_2+2$	$\|l_1,-1_1+2>\|l_{2,},-l_2>$	$\|l_1,-1_1+1>\|l_{2,},-l_2+1>$	$\|l_1,-1_1>\|l_{2,},-l_2+2>$	..3
$M=-l_1-l_2+1$	$\|l_1,-1_1+1>\|l_{2,},-l_2>$	$\|l_1,-1_1>\|l_{2,},-l_2+1>$		…2
$M=-l_1-l_2$	$\|l_1,-1_1>\|l_{2,},-l_2>$			…1
# functions	$2(l_1+l_2)+1$	$2(l_1+l_2-1)+1$	$2(l_1+l_2-2)+1$	

Example of how to get Clebsch Gordon coefficients

$s_1 = ½$ $m_{s1} = ±½$
$s_2 = ½$ $m_{s2} = ±½$

a) Uncoupled wave functions:

$$| s_1, m_{s_1}; s_2, m_{s_2} >$$

$$= | \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} >= \alpha_1 \alpha_2$$

$$= | \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} >= \beta_1 \alpha_2$$

$$= | \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2} >= \alpha_1 \beta_2$$

$$= | \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} >= \beta_1 \beta_2$$

b.) Clebsch-Gordon Series: $S = (s_1 + s_2) \rightarrow |s_1 - s_2| = \left(\frac{1}{2} + \frac{1}{2}\right) - \left|\frac{1}{2} - \frac{1}{2}\right| = 1{,}0$

c.) Coupled wave functions:

$$| s_1, s_2, S, M_S >$$

$$= | \frac{1}{2}, \frac{1}{2}, 0, 0 >$$

Singlet state, $S = 0$; $M_S = 0$

$$= | \frac{1}{2}, \frac{1}{2}, 1, 1 >$$

$$= | \frac{1}{2}, \frac{1}{2}, 1, 0 >$$

$$= | \frac{1}{2}, \frac{1}{2}, 1, -1 >$$

Triplet State, $S=1$; $M_S=1,0,-1$

d) Start with **triplet state**

$$|\frac{1}{2},\frac{1}{2},1,1> \quad \text{can only be formed from}$$

$$|s_1, m_{s_1}; s_2, m_{s_2}> = |\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}> \because M_S = m_{s_1} + m_{s_2}$$

$$\Rightarrow |\frac{1}{2},\frac{1}{2},1,1> = |\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}> \Rightarrow C_{m_{s_1},m_{s_1}} = \boxed{C_{\frac{1}{2},\frac{1}{2}} = 1}$$

Similarly, $|\frac{1}{2},\frac{1}{2},1,-1>$ can only be formed from

$$|s_1, m_{s_1}; s_2, m_{s_2}> = |\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}>$$

$$\Rightarrow |\frac{1}{2},\frac{1}{2},1,-1> = |\frac{1}{2},-\frac{1}{2};\frac{1}{2},-\frac{1}{2}> \Rightarrow C_{m_{s_1},m_{s_1}} = \boxed{C_{-\frac{1}{2},-\frac{1}{2}} = 1}$$

To get $|\frac{1}{2},\frac{1}{2},1,0>$ apply either lowering operator to $|\frac{1}{2},\frac{1}{2},1,1>$

or raising operator to $|\frac{1}{2},\frac{1}{2},1,-1>$

Use lowering operator. Recall:

$$\hat{S}_- = \hat{s}_{1-} + \hat{s}_{2-} \quad \text{and} \quad \hat{S}_- |S, M_S> = \sqrt{S(S+1) - M_S(M_S - 1)}\hbar |S, M_S - 1>$$

$$\therefore \hat{S}_- |\frac{1}{2},\frac{1}{2},1,1> = (\hat{s}_{1-} + \hat{s}_{2-})|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}>$$

Left-hand side: $\hat{S}_- |\frac{1}{2},\frac{1}{2},1,1> = \sqrt{(1)(2)-(1)(0)}\hbar |\frac{1}{2},\frac{1}{2},1,0> = \sqrt{2}\hbar |\frac{1}{2},\frac{1}{2},1,0>$

Right-hand side: $(\hat{s}_{1-} + \hat{s}_{2-})|\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}> = \hat{s}_1 |\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}> + \hat{s}_2 |\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}>$

$$= \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\hbar |\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}> + \sqrt{\frac{1}{2}\left(\frac{1}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\hbar |\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>$$

$$= \sqrt{\frac{3}{4}+\frac{1}{4}}\hbar |\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}> + \sqrt{\frac{3}{4}+\frac{1}{4}}\hbar |\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>$$

$$= \hbar |\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}> + \hbar |\frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2}>$$

$$\therefore \sqrt{2}\hbar \,|\, \frac{1}{2},\frac{1}{2},1,0 >= \hbar \,|\, \frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2} > +\hbar \,|\, \frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2} >$$

$$\Rightarrow |\, \frac{1}{2},\frac{1}{2},1,0 >= \frac{1}{\sqrt{2}} \,|\, \frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2} > + \frac{1}{\sqrt{2}} \,|\, \frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2} >$$

$$\Rightarrow \boxed{C_{-\frac{1}{2},\frac{1}{2}} = \frac{1}{\sqrt{2}}} \qquad \boxed{C_{\frac{1}{2},-\frac{1}{2}} = \frac{1}{\sqrt{2}}}$$

For the **singlet state**

Let $|\, \frac{1}{2},\frac{1}{2},0,0 >= a_1 \,|\, \frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2} > +a_2 \,|\, \frac{1}{2},\frac{1}{2};\frac{1}{2},-\frac{1}{2} >$ and require

$$\left\langle \frac{1}{2},\frac{1}{2},0,0 \middle| \frac{1}{2},\frac{1}{2},0,0 \right\rangle = 1 \quad \text{and} \quad \left\langle \frac{1}{2},\frac{1}{2},0,0 \middle| \frac{1}{2},\frac{1}{2},1,0 \right\rangle = 0$$