2.3: A Digression: Matrix Algebra

1.) Matrices

A matrix is a grouping of components (numbers) in rows and columns.

$$\tilde{A} = \begin{array}{rl} & \begin{array}{ccccc} \text{column 1} & & & & \\ \downarrow & & & & \end{array} \\ \text{row 1} \rightarrow & \begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} \end{pmatrix} \end{array} = \{a_{ij}\}$$

=m x n matrix

Rules: $$\tilde{A}\cdot\tilde{B}=\{a_{ij}\}\cdot\{b_{ij}\}=\sum_k a_{ik}b_{kj}$$

For example: $$\tilde{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \tilde{B}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\tilde{A}\cdot\tilde{B}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\cdot\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$$

A numerical example: $$\tilde{A}=\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \qquad \tilde{B}=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$$\tilde{A}\cdot\tilde{B}=\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}=\begin{pmatrix} 2+6 & 6+12 \\ 1+10 & 3+20 \end{pmatrix}=\begin{pmatrix} 8 & 18 \\ 11 & 23 \end{pmatrix}$$

$$\tilde{B}\cdot\tilde{A}=\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}=\begin{pmatrix} 2+3 & 3+15 \\ 4+4 & 6+20 \end{pmatrix}=\begin{pmatrix} 5 & 18 \\ 8 & 26 \end{pmatrix}$$

In general: $\widetilde{A}\cdot\widetilde{B}\neq\widetilde{B}\cdot\widetilde{A}$ that is; matrix multiplication is not **commutative**.

If $\widetilde{A}\cdot\widetilde{B}=\widetilde{B}\cdot\widetilde{A}\Rightarrow\left[\widetilde{A},\widetilde{B}\right]=0$ then we say **A** and **B** commute.

Matrix multiplication is not restricted to square matrices; that is, n x n

$$\Rightarrow \widetilde{A}(m\,x\,\boxed{n})\cdot\widetilde{B}(\boxed{n}\,x\,p)\rightarrow(m\,x\,p)\text{matrix}$$

common = necessary condition

e.g.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}=\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{13}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \end{pmatrix}$$

(2 x 3) (3 x 2) (2 x 2)

Further properties involving matrix multiplication:

$$\left(\tilde{A}\,\tilde{B}\,\right)^{T} = \tilde{B}\,\tilde{A}$$

$$\left(\tilde{A}\,\tilde{B}\,\right)^{+} = \tilde{B}^{\,+}\,\tilde{A}^{\,+}$$

$$\left(\tilde{A}\,\tilde{B}\,\right)^{*} = \tilde{A}^{\,*}\,\tilde{B}^{\,*}$$

$$\left(\tilde{A}\,\tilde{B}\,\right)^{-1} = \tilde{B}^{\,-1}\,\tilde{A}^{\,-1}$$

Distribution: $\tilde{A}\left(\tilde{B}+\tilde{C}\right)=\tilde{A}\tilde{B}+\tilde{A}\tilde{C}$

Association: $\tilde{A}\left(\tilde{B}\tilde{C}\right)=\left(\tilde{A}\tilde{B}\right)\tilde{C}$

Matrix addition and subtraction:

$$\widetilde{A} \pm \widetilde{B} = \left\{a_{ij} \pm b_{ij}\right\}$$

Transpose: $\widetilde{A}^T = \left\{a_{ij}\right\}^T = \left\{a_{ji}\right\}$

e.g. $$\begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$$

Complex Conjugate: $\widetilde{A}^* = \left\{a_{ij}\right\}^* = \left\{a_{ij}^*\right\}$

Self-adjoint: $\widetilde{A}^+ = \left(\widetilde{A}^T\right)^* = \left\{a_{ij}\right\}^+ = \left\{a_{ji}^*\right\}$

Leads to the concept of **Hermitian** matrix: $\widetilde{A}^+ = \widetilde{A} \Rightarrow \left\{a_{ij}\right\} = \left\{a_{ji}^*\right\}$

This lead to real diagonal matrix elements.

Therefore, if $\mathbf{A}^+ = \mathbf{A}$ then **A** corresponds to a physical quantity.

$$\Rightarrow A_{ii} = \left\langle \psi_i \mid \hat{A} \mid \psi_i \right\rangle$$ = a value of a **physical property** in a state Ψ_i

If $$\underset{\sim}{A} = \begin{pmatrix} a & e-if \\ c-id & b \end{pmatrix} \qquad \underset{\sim}{A}^+ = \begin{pmatrix} a & c+id \\ e+if & b \end{pmatrix}$$

then **A** is self-adjoint if c = e and d = -f.

Further definitions involving matrices:

A matrix **A** is:

symmetric if $\tilde{A} = \tilde{A}^T$

skew – symmetric if $\tilde{A} = -\tilde{A}^T$

real if $\tilde{A} = \tilde{A}^*$

pure imaginary if $\tilde{A} = -\tilde{A}^*$

Hermitian if $\tilde{A} = \tilde{A}^+$

skew Hermitian if $\tilde{A} = -\tilde{A}^+$

orthogonal if $\tilde{A}^{-1} = \tilde{A}^T$

unitary if $\tilde{A}^{-1} = \tilde{A}^+$