Midterm Solutions C474b 2007

1) $|\psi\rangle = \sqrt{\frac{3}{8}}\,Y_{1,1} + \sqrt{\frac{1}{8}}\,Y_{1,0} + A\,Y_{1,-1}$

a) Normalize $|\psi\rangle \Rightarrow \langle\psi|\psi\rangle = 1$

$= \langle \sqrt{\tfrac{3}{8}}\,Y_{1,1} + \sqrt{\tfrac{1}{8}}\,Y_{1,0} + A\,Y_{1,-1} | \sqrt{\tfrac{3}{8}}\,Y_{1,1} + \sqrt{\tfrac{1}{8}}\,Y_{1,0} + A\,Y_{1,-1} \rangle$

$= \frac{3}{8}\underbrace{\langle Y_{1,1}|Y_{1,1}\rangle}_{=1} + \frac{1}{8}\underbrace{\langle Y_{1,0}|Y_{1,0}\rangle}_{=1} + A^2\underbrace{\langle Y_{1,-1}|Y_{1,-1}\rangle}_{=1} + \text{terms} \propto \underbrace{\langle Y_{\ell,m}|Y_{\ell',m'}\rangle}_{=0}$

$\Rightarrow 1 = \frac{3}{8} + \frac{1}{8} + A^2 \Rightarrow A^2 = 1 - \frac{1}{2} = \frac{1}{2} \Rightarrow A = \sqrt{\frac{1}{2}}$

b) $\langle\psi|\hat{L}_z|\psi\rangle = \langle \sqrt{\tfrac{3}{8}}\,Y_{1,1} + \sqrt{\tfrac{1}{8}}\,Y_{1,0} + \sqrt{\tfrac{1}{2}}\,Y_{1,-1} | \hat{L}_z | \sqrt{\tfrac{3}{8}}\,Y_{1,1} + \sqrt{\tfrac{1}{8}}\,Y_{1,0} + \sqrt{\tfrac{1}{2}}\,Y_{1,-1}\rangle$

$= \frac{3}{8}\langle Y_{1,1}|\hat{L}_z|Y_{1,1}\rangle + \frac{1}{8}\langle Y_{1,0}|\hat{L}_z|Y_{1,0}\rangle + \frac{1}{2}\langle Y_{1,-1}|\hat{L}_z|Y_{1,-1}\rangle + \text{terms} \propto \underbrace{\langle Y_{\ell,m}|\hat{L}_z|Y_{\ell',m'}\rangle}_{=0}$

$= \frac{3\hbar}{8}\underbrace{\langle Y_{1,1}|Y_{1,1}\rangle}_{=1} + \frac{0\hbar}{8}\underbrace{\langle Y_{1,0}|Y_{1,0}\rangle}_{=1} + \frac{-\hbar}{2}\underbrace{\langle Y_{1,-1}|Y_{1,-1}\rangle}_{=1}$

$= \boxed{-\frac{\hbar}{8}}$

Similarly $\langle\psi|\hat{L}^2|\psi\rangle = \frac{3}{8}\langle Y_{1,1}|\hat{L}^2|Y_{1,1}\rangle + \frac{1}{8}\langle Y_{1,0}|\hat{L}^2|Y_{1,0}\rangle + \frac{1}{2}\langle Y_{1,-1}|\hat{L}^2|Y_{1,-1}\rangle$

$= \frac{3}{8}(1)(2)\hbar^2 + \frac{1}{8}(1)(2)\hbar^2 + \frac{1}{2}(1)(2)\hbar^2 = \hbar^2\left[\frac{3}{4} + \frac{1}{4} + 1\right] = \boxed{2\hbar^2}$

c) Probability $= |\langle Y_{1,0}|\psi\rangle|^2 = \frac{1}{8}$

② a) $$\tilde{A}\tilde{B} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 2+0+1 & 1+0+(-1) & 1+0+2 \\ 0 & 0 & 0 \\ 2 & 1+0+0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\tilde{B}\tilde{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2+0+1 & 0 & 2 \\ 1+0-1 & 0 & 1 \\ 1+0+2 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 & 2 \\ 0 & 0 & 1 \\ 3 & 0 & 1 \end{pmatrix}$$ $\tilde{A}\tilde{B} \neq \tilde{B}\tilde{A}$ matrices do <u>not</u> commute

b) a matrix is Hermitian if $\tilde{A} = (\tilde{A}^T)^*$ etc

$$\tilde{A} = \begin{pmatrix} 9 & 1 \\ 3 & 4 \end{pmatrix} \Rightarrow (\tilde{A}^T)^* = \begin{pmatrix} 9 & 3 \\ 1 & 4 \end{pmatrix}^* = \begin{pmatrix} 9 & 3 \\ 1 & 4 \end{pmatrix}$$ NOT Hermitian.

$$\tilde{B} = \begin{pmatrix} 9i & 4+4i \\ -4+4i & 7i \end{pmatrix} \Rightarrow (\tilde{B}^T)^* = \begin{pmatrix} 9i & -4+4i \\ 4+4i & 7i \end{pmatrix}^* = \begin{pmatrix} -9i & -4-4i \\ 4-4i & -7i \end{pmatrix}$$ NOT Hermitian.

$$\tilde{C} = \begin{pmatrix} 9 & 4-4i \\ 4+4i & 7 \end{pmatrix} \Rightarrow (\tilde{C}^T)^* = \begin{pmatrix} 9 & 4+4i \\ 4-4i & 7 \end{pmatrix}^* = \begin{pmatrix} 9 & 4-4i \\ 4+4i & 7 \end{pmatrix}$$ Hermitian.

c) To find eigenvalues solve $|\tilde{A} - \lambda\tilde{I}| = 0$

$$\therefore \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow -(3-\lambda)(3+\lambda) - 16 = 0$$

$$\therefore (3-\lambda)(3+\lambda) + 16 = 0 \Rightarrow \lambda^2 - 25 = 0 \Rightarrow \boxed{\lambda = \pm 5}$$

$$9 \quad \lambda^2 - 3\lambda + 3\lambda + 16 = 0$$

When $\underline{\lambda = 5}$ $\Rightarrow \begin{pmatrix} 3-5 & 4 \\ 4 & -3-5 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\therefore\ -2c_1 + 4c_2 = 0 \Rightarrow -c_1 + 2c_2 = 0 \quad \therefore c_2 = \frac{c_1}{2}$

$\therefore$ eigenvector $\equiv \tilde{c} = c_1 \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$ $\qquad \tilde{c}^{+}\tilde{c} = 1$ for normalization

$\therefore c_1^2 \ \underbrace{1 \ \ \tfrac{1}{2}} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = 1 \Rightarrow c_1^2 \cdot (1 + \frac{1}{4}) = 1 \Rightarrow c_1^2 \frac{5}{4} = 1 \Rightarrow c_1 = \frac{2}{\sqrt{5}}$

$\Rightarrow c_2 = \frac{1}{\sqrt{5}} \qquad \therefore \tilde{c} = \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}.$

When $\lambda = -5 \Rightarrow \begin{pmatrix} 3+5 & 4 \\ 4 & -3+5 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\therefore\ 4b_1 + 2b_2 = 0 \Rightarrow b_1 + \frac{1}{2}b_2 = 0 \quad \therefore b_2 = -2b_1$

$\tilde{b}^{+}\tilde{b} = 1 \Rightarrow b_1^2 \ \underbrace{1 \ \ -2} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 \Rightarrow b_1^2(1+4) = 1 \Rightarrow b_1^2 = \frac{1}{5} \quad \therefore b_1 = \frac{1}{\sqrt{5}}$

$\therefore b_2 = -\frac{2}{\sqrt{5}} \qquad \Rightarrow \tilde{b} = \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$

note $\tilde{b}^{+}\tilde{c} = \underbrace{1/\sqrt{5} \ \ -2/\sqrt{5}} \begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} = 0$ $\qquad$ ie eigenvectors are normalized and orthogonal.

(3) $E_3^{(0)} = \frac{5}{2}\hbar\omega_0$

$E_3^{(1)} = \frac{1}{2}\varepsilon\hbar\omega_0$ read off as $H_{33}^{(1)}$ matrix element

$$E_3^{(2)} = -\sum_{k\neq 3} \frac{|H_{k3}^{(1)}|^2}{E_k^{(0)} - E_3^{(0)}} = \sum_{k\neq 3} \frac{|H_{k3}^{(1)}|^2}{E_3^{(0)} - E_k^{(0)}}$$

$$= \frac{|H_{13}^{(1)}|^2}{E_3^{(0)} - E_1^{(0)}} + \frac{|H_{23}^{(1)}|^2}{E_3^{(0)} - E_2^{(0)}} + \frac{|H_{43}^{(1)}|^2}{E_3^{(0)} - E_4^{(0)}}$$

$$= \frac{\left|-\frac{\varepsilon\hbar\omega_0}{\sqrt{2}}\right|^2}{\frac{5\hbar\omega_0}{2} - \frac{\hbar\omega_0}{2}} + 0 + 0 = \frac{\frac{\varepsilon^2\hbar^2\omega_0^2}{2}}{2\hbar\omega_0} = +\frac{\varepsilon^2\hbar\omega_0}{4}$$

$$\therefore E_3 = E_3^{(0)} + E_3^{(1)} + E_3^{(2)} = \frac{5}{2}\hbar\omega_0 + \frac{1}{2}\varepsilon\hbar\omega_0 + \frac{\varepsilon^2}{4}\hbar\omega_0$$

$$\boxed{E_3 = \hbar\omega_0\left[\frac{5}{2} + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{4}\right]}$$

$$\psi_3^{(1)} = -\sum_{k\neq 3} \frac{H_{k3}^{(1)}\psi_k^{(0)}}{E_k^{(0)} - E_3^{(0)}} = -\frac{H_{13}^{(1)}\psi_1^{(0)}}{E_1^{(0)} - E_3^{(0)}} - \frac{H_{23}^{(1)}\psi_2^{(0)}}{E_2^{(0)} - E_3^{(0)}} - \frac{H_{43}^{(1)}\psi_4^{(0)}}{E_4^{(0)} - E_3^{(0)}}$$

$$= \frac{+\frac{\varepsilon\hbar\omega_0}{\sqrt{2}}}{\frac{\hbar\omega_0}{2} - \frac{5\hbar\omega_0}{2}}\psi_1^{(0)} - 0 - 0 = -\frac{\varepsilon}{2\sqrt{2}} \cdot \frac{\hbar\omega_0}{\hbar\omega_0}\psi_1^{(0)} = -\frac{\varepsilon}{2\sqrt{2}}\psi_1^{(0)}$$

$$\therefore \psi_3 = \psi_3^{(0)} + \psi_3^{(1)} = \boxed{\psi_3^{(0)} - \frac{\varepsilon}{2\sqrt{2}}\psi_1^{(0)}}$$

(2) Secular determinant $|H_{qj}^{(1)} - E_j^{(1)}\delta_{ij}| = 0$ for this

a) problem is
$$\begin{vmatrix} -E^{(1)} & \varepsilon \\ \varepsilon & -E^{(1)} \end{vmatrix} = 0$$

$$\therefore \quad E^{(1)2} - \varepsilon^2 = 0 \quad \Rightarrow \quad E^{(1)} = \pm \varepsilon$$

b) $\underline{E^{(1)} = +\varepsilon}$ $\Rightarrow \begin{pmatrix} -\varepsilon & \varepsilon \\ \varepsilon & -\varepsilon \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\therefore \quad -c_1\varepsilon + c_2\varepsilon = 0 \quad \Rightarrow \quad -c_1 + c_2 = 0 \quad \Rightarrow c_2 = +c_1$$

$$\therefore \quad \tilde{c}^{+}\tilde{c} = 1 = c_1^2(1 \quad +1)\begin{pmatrix} 1 \\ +1 \end{pmatrix} = 2c_1^2 = 1 \Rightarrow c_1 = \frac{1}{\sqrt{2}} \; ; c_2 \; +\frac{1}{\sqrt{2}}$$

$$\therefore \quad \boxed{\phi_1^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

$\underline{E^{(1)} = -\varepsilon}$ $\Rightarrow \begin{pmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow c_2 = -c_1$

$$\therefore \quad \tilde{c}^{+}\tilde{c} = 1 = c_1^2(1 \quad -1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 2c_1^2 = 1 \Rightarrow c_1 = \frac{1}{\sqrt{2}} \quad c_2 = -\frac{1}{\sqrt{2}}$$

$$\therefore \quad \boxed{\phi_2^{(0)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

c) $$E_q^{(2)} = -\sum_{k \neq q} \frac{|\langle \psi_k^{(0)} | \hat{H}^{(1)} | \phi_q^{(0)} \rangle|^2}{E_k^{(0)} - E_q^{(0)}}$$

$$\therefore \quad E_1^{(2)} = -\frac{\langle \psi_3^{(0)} | \hat{H}^{(1)} | \phi_1^{(0)} \rangle}{E_3^{(0)} - E_1^{(0)}} = -\frac{|\langle \psi_3^{(0)} | \hat{H}^{(1)} | \frac{1}{\sqrt{2}}\psi_1^{(0)} + \frac{1}{\sqrt{2}}\psi_2^{(0)} \rangle|^2}{2 - 1}$$

$$= -\frac{\left|\frac{1}{\sqrt{2}}H_{31}^{(1)} + \frac{1}{\sqrt{2}}H_{32}^{(1)}\right|^2}{1} = -\left|\frac{1}{\sqrt{2}}\cdot 0 + \frac{1}{\sqrt{2}}\cdot \varepsilon\right|^2 = -\frac{\varepsilon^2}{2}$$

$$E_2^{(2)} = -\frac{\left|\langle \psi_3^{(0)} | \hat{H}^{(1)} | \frac{1}{\sqrt{2}}\psi_1^{(0)} - \frac{1}{\sqrt{2}}\psi_2^{(0)} \rangle\right|^2}{2-1} = -\left|\frac{1}{\sqrt{2}}H_{31}^{(1)} - \frac{1}{\sqrt{2}}H_{32}^{(1)}\right|^2$$

$$= -\left|0 - \frac{1}{\sqrt{2}}\varepsilon\right|^2 = -\frac{\varepsilon^2}{2}$$

$$\therefore E_1 = E_1^{(0)} + E_1^{(1)} + E_1^{(2)} = 1 + \varepsilon - \frac{\varepsilon^2}{2}$$

$$E_2 = E_2^{(0)} + E_2^{(1)} + E_2^{(2)} = 1 - \varepsilon - \frac{\varepsilon^2}{2}.$$

(5) $$P_q(t) = \frac{4\,|H_{jq}^{(1)} H_{qj}^{(1)}|}{\hbar^2} \frac{\sin^2\left(\frac{1}{2}(\omega_{qj} - \omega)t\right)}{(\omega_{qj} - \omega)^2}$$

– the $\frac{\sin^2\left(\frac{1}{2}(\omega_{qj} - \omega)t\right)}{(\omega_{qj} - \omega)^2}$ term incorporates resonance; that is, when the light frequency ω = frequency of energy level separation $\omega_{qj} = \frac{E_q^{(0)} - E_j^{(0)}}{\hbar}$

$H_{qj}^{(1)}$ = electric dipole transition matrix element connecting states q and j; that is $H_{qj}^{(1)} = \langle \psi_q^{(0)} | \hat{H}^{(1)} | \psi_j^{(0)} \rangle = -\langle \psi_q^{(0)} | \hat{\mu} | \psi_j^{(0)} \rangle E$

$$= -eE \langle \psi_q^{(0)} | \vec{r} | \psi_j^{(0)} \rangle$$

– the conditions under which $H_{qj}^{(1)} \neq 0$ correspond to selection rules.

– t = time, $\hbar = \frac{h}{2\pi} \equiv \frac{\text{Planck's constant}}{2\pi}$.