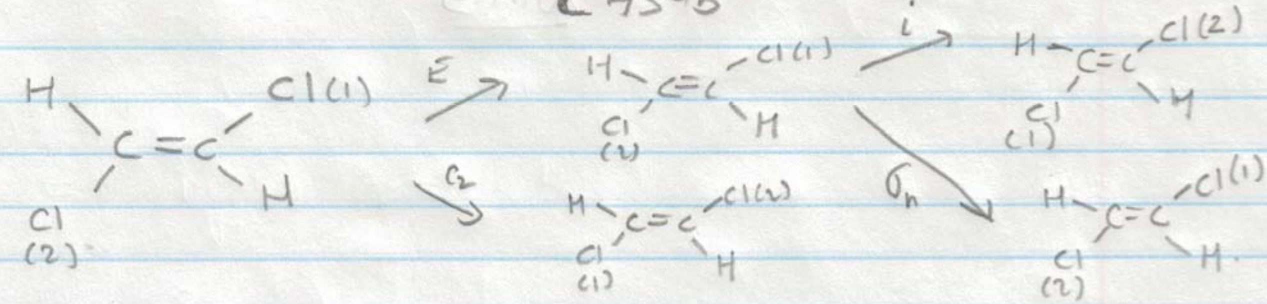
Solutions Problem Set 3 2008

C734b

1. 

a) $\langle Cl(1)\ Cl(2) | \Gamma(E) = \langle Cl(1)\ Cl(2) |$

$\therefore \Gamma(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \chi = 2$

$\langle Cl(1)\ Cl(2) | \Gamma(C_2) = \langle Cl(2), Cl(1) |$

$\therefore \Gamma(C_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \chi = 0$

$\langle Cl(1)\ Cl(2) | \Gamma(i) = \langle Cl(2)\ Cl(1) |$

$\therefore \Gamma(i) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \chi = 0$

$\langle Cl(1)\ Cl(2) | \Gamma(\sigma_h) = \langle Cl(1)\ Cl(2) |$

$\therefore \Gamma(\sigma_h) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad \chi = 2$

if irreducible $\sum_T |\chi(T)|^2 = h = 4$

but $\sum_T |\chi(T)|^2 = 4 + 0 + 0 + 4 = 8 > 4$ $\therefore$ reducible.

b) given $\{1, 1, 1, 1\} \Rightarrow \chi(E) = 1, \chi(C_2) = 1, \chi(i) = 1$

$\therefore \boxed{A_g}$

$\{1, -1, -1, 1\} \Rightarrow \chi(E) = 1, \chi(C_2) = -1, \chi(i) = 1 \therefore \boxed{B_u}$

c) $\langle z | \Gamma(E) = +\langle z|$ $\quad \therefore \Gamma(E) = (1) \quad \chi = 1$

$\langle z | \Gamma(C_2) = +\langle z|$ $\quad \therefore \Gamma(C_2) = (1) \quad \chi = 1$

$\langle z | \Gamma(i) = -\langle z|$ $\quad \therefore \Gamma(i) = (-1) \quad \chi = -1$

$\langle z | \Gamma(\sigma_h) = -\langle z|$ $\quad \therefore \Gamma(\sigma_h) = (-1) \quad \chi = -1$

$\{1 \quad 1 \quad -1 \quad -1\}$ $\quad \chi(E) = 1, \ \chi(C_2) = 1 \quad \chi(i) = -1$

$\therefore \boxed{A_u}$

d) Here we can use orthogonality of IRs

$$\sum_T |\chi_4(T)|^2 = 4$$

and $\sum_{IR} l_{IR}^2 = 4 \quad \therefore (1)^2 + (1)^2 + (1)^2 + l_4^2 = 4$

$\therefore l_4^2 = 4 - 3 = 1 \quad = 1$

$\boxed{l_4 = 1}$

$\therefore \chi_4(E) = 1$

$\therefore (1)^2 + \chi_4(C_2)^2 + \chi_4(i)^2 + \chi_4(\sigma_h)^2 = 4$

or $\chi_4(C_2)^2 + \chi_4(i)^2 + \chi_4(\sigma_h)^2 = 3 \quad (1)$

$\chi_1(E)\chi_4(E) + \chi_4(C_2)(1) + \chi_4(i)(1) + \chi_4(\sigma_h)(1) = 0$

or $\chi_4(C_2) + \chi_4(i) + \chi_4(\sigma_h) = -1 \quad (2)$

Similarly: $\chi_2(E)\chi_4(E) + \chi_4(C_2)(-1) + \chi_4(i)(-1) + \chi_4(\sigma_h)(1) = 0$

or $-\chi_4(C_2) - \chi_4(i) + \chi_4(\sigma_h) = -1 \quad (3)$

add (2)+(3) $\quad 2\chi_4(\sigma_h) = -2 \quad \therefore \chi_4(\sigma_h) = -1$

subtract (2)−(3) $\quad 2\chi_4(C_2) + 2\chi_4(i) = 0$

$$\therefore \chi_4(i) = -\chi_4(C_2)$$

Into (1) $\Rightarrow 2\chi_4(i)^2 + (-1)^2 = 3$

$\chi_4(i) = 1$

$\therefore \chi_4(C_2) = -1 \Rightarrow \boxed{B_g}$

Character Table:

C_{2h}	E	C_2	i	σ_h
A_g	1	1	1	1
B_g	1	−1	1	−1
A_u	1	1	−1	−1
B_u	1	−1	−1	−1

(2.) $\langle x, y, z | \Gamma_m(E) = \langle x\ y\ z |$

$\therefore \Gamma_m(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad \chi = 3$

$\langle x, y, z | \Gamma_m(C_2) = \langle -x, -y, z |$

$\therefore \Gamma_m(C_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad \chi = -1$

$\langle x, y, z | \Gamma_m(i) = \langle -x, -y, -z |$

$\therefore \Gamma_m(i) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\qquad \chi = -3$

$\langle x, y, z | \Gamma_m(\sigma_h) = \langle x, y, -z |$

$\therefore \Gamma_m(\sigma_h) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ $\qquad \chi = +1$

character set $= \{3, -1, -3, 1\}$. (Clearly reducible!)

b) use $a_j = \frac{1}{h} \sum_R \chi_m(R)\, \chi_j(R)$ $\qquad h = 4$

$a_{A_g} = \frac{1}{4}[(3)(1) + (-1)(1) + (-3)(1) + (1)(1)] = 0$

$$a_{B_g} = \frac{1}{4}\left[(3)(1) + (-1)(-1) + (-3)(1) + (1)(-1)\right] = 0$$

$$a_{A_u} = \frac{1}{4}\left[(3)(1) + (-1)(1) + (-3)(-1) + (1)(-1)\right] = 1$$

$$a_{B_u} = 1/4\left[(3)(1) + (-1)(-1) + (-3)(-1) + (1)(1)\right] = 2$$

$$\therefore \Gamma_m = A_u \oplus 2B_u$$

c) Group Multiplication Table

C_{2h}	E	C_2	i	σ_h
E	EE	C_2E	iE	$\sigma_h E$
C_2	EC_2	C_2C_2	iC_2	$\sigma_h C_2$
i	Ei	$C_2 i$	ii	$\sigma_h i$
σ_h	$E\sigma_h$	$C_2\sigma_h$	$i\sigma_h$	$\sigma_h\sigma_h$

$$EE = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = E$$

$$EC_2 = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix} \quad C_2$$

$$= C_2E$$

$$Ei = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix} \quad i$$

$$= iE$$

$$E\sigma_h = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = \begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix} = \sigma_h$$

$$= \sigma_h E$$

$$C_2 C_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

$$i i = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

$$\sigma_h \sigma_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = E$$

$$C_2 i = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \sigma_h$$

$$= i C_2$$

$$C_2 \sigma_h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i$$

$$= \sigma_h C_2$$

$$i \sigma_h = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = C_2$$

$$= \sigma_h i$$

Note all matrices are diagonal $\Rightarrow$ group is Abelian.

Table is

	E	C_2	i	σ_h
E	E	C_2	i	σ_h
C_2	C_2	E	σ_h	i
i	i	σ_h	E	C_2
σ_h	σ_h	i	C_2	E

obeys rearrangement theorem.