Chapter 2 - Free Vibration of Multi-Degree-of-Freedom Systems - II

We can obtain an approximate solution to the fundamental natural frequency through an approximate formula developed using energy principles by Lord Rayleigh. As with single-degree-of-freedom systems, MDOF systems can also use this approximation:

$$\omega_1^2 = g\frac{\sum_i Q_i u_i}{\sum_i Q_i u_i^2} = g\frac{\sum_i m_i u_i}{\sum_i m_i u_i^2}$$

where u_i = the static deflection under the dead load of the structure Q_i, acting in the direction of motion, and g = the acceleration due to gravity. Thus, the first mode is approximated in shape by the static deflection under dead load. For a building, this can be applied to each of the X and Y directions to obtain the estimates of the fundamental sway modes.

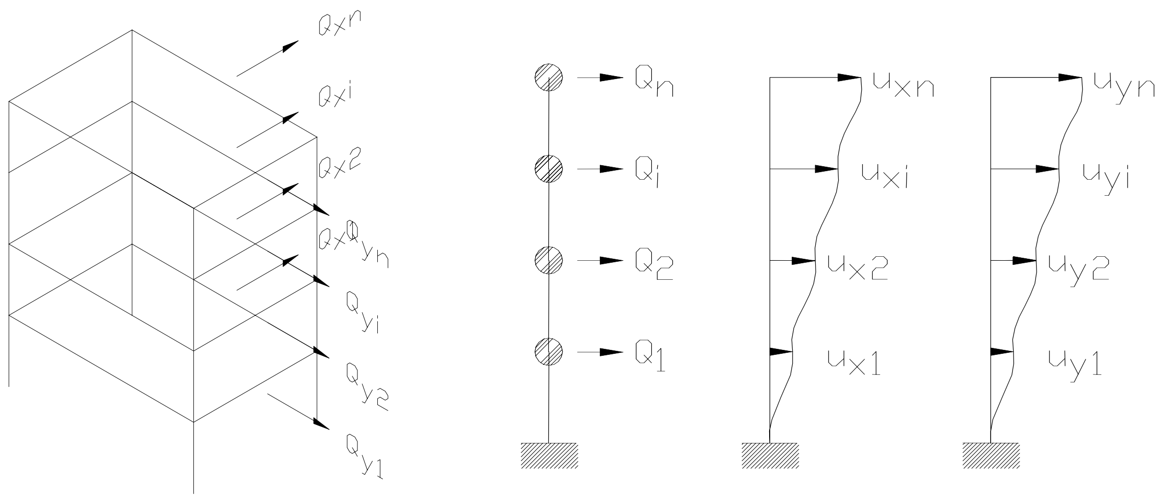


Fig. 2.1a) Deflection for Rayleigh's Formula Applied to Buildings

Likewise for a bridge, by applying the dead load in each of the vertical and horizontal directions, the fundamental lift and drag modes can be obtained. The torsional mode can also be approximated by applying the dead load at the appropriate radius of gyration and determining the resulting rotation angle.

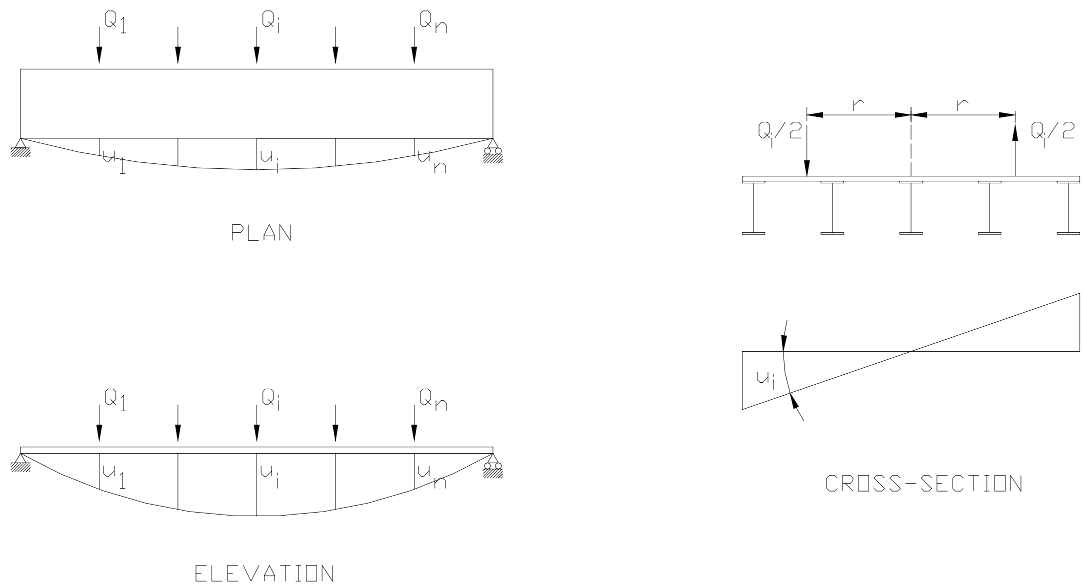
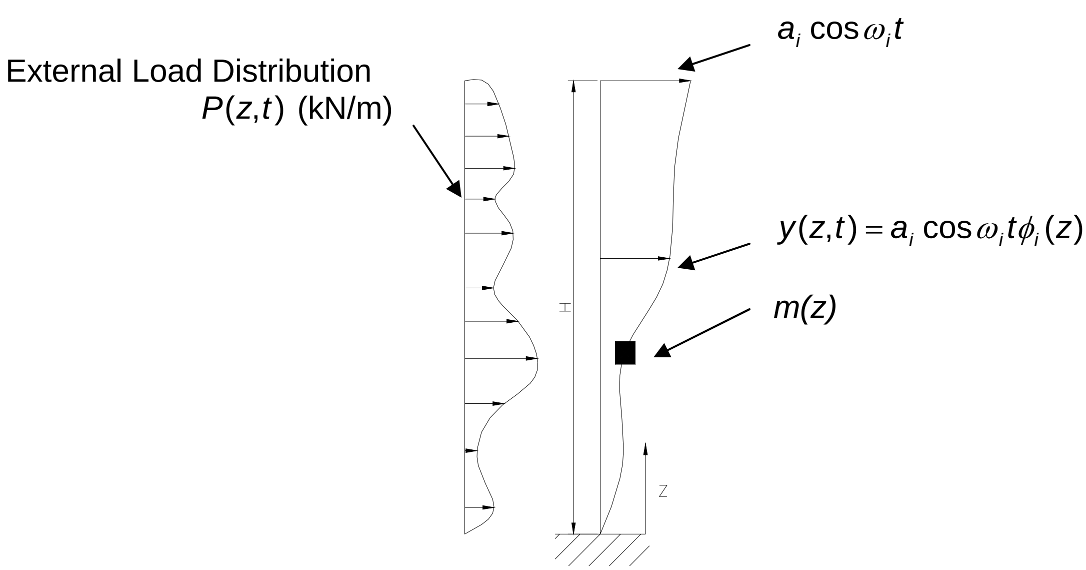


Fig. 2.1b) Deflection for Rayleigh's Formula Applied to Bridges

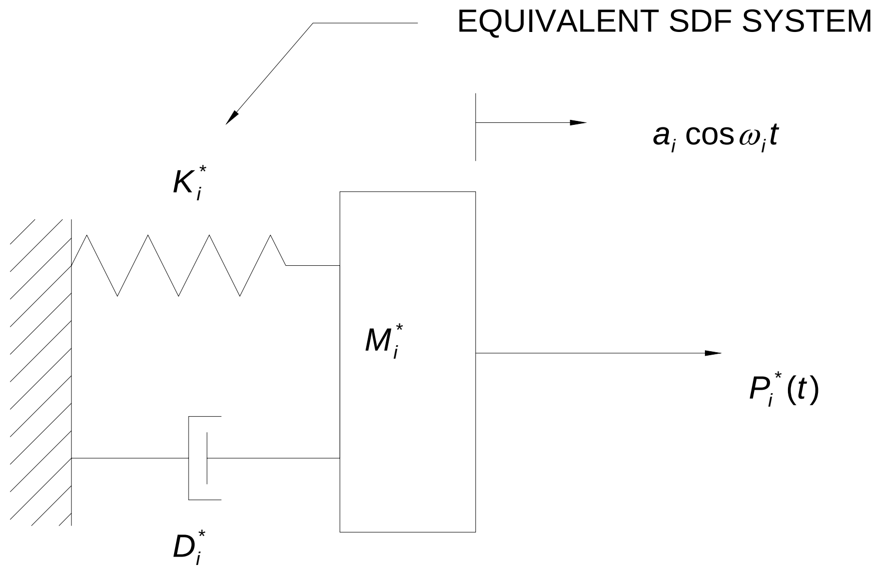
Even when performing a detailed dynamic analysis using computer software like SAP, ANSYS or ALGOR, a check using Rayleigh's method is advisable. Often, for most preliminary designs, a detailed dynamic analysis is not required and a first-order analysis using Rayleigh's method is all that is required.

<u>Generalized Coordinates</u>

Generalized coordinates are a means of simplification of a multi-degree-of-freedom system into a series of equivalent single-degree-of-freedom systems.



Continuous Structures can be idealized mode-by-mode in *Generalized Parameters* :



Generalized Mass:

$$M_i^* = \int_0^H m(z)\phi_i^2(z)dz$$

Generalized Stiffness:

$$K_i^* = \omega_i^2 M_i^*$$
$$D_i^* = D_i = \text{not generalized}$$
$$\omega_i^* = \omega_i = \text{not generalized}$$

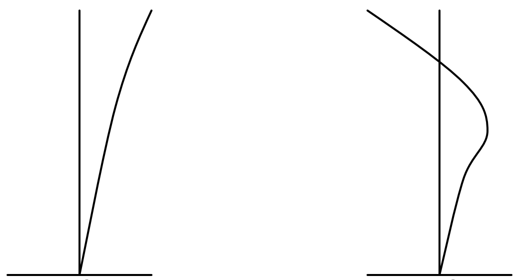
Generalized Force:

$$P_i^*(t) = \int_0^H P(z,t)\phi_i(z)dz$$

The response of the actual structure in mode "*i*" is the same as that of its equivalent SDF system in mode "*i*" when defined by its Generalized Properties – Stiffness, Mass and Force.

Orthogonality of Modes

Orthogonality of modes is a very important relationship between any two modes of free vibration. It means that each mode is truly independent of another.



Mode 1 Mode 2

Recall that we found that the natural frequencies ω_j and corresponding modes can be determined algebraically:

$$\left([k]-\omega_j^2[m]\right)\{u_j\}=0, \quad \text{where } \{u_j\}=\begin{Bmatrix} u_1(j) \\ u_2(j) \\ . \\ . \\ u_n(j) \end{Bmatrix} \text{ the } \textit{Eigenvectors} \text{ or } \textit{Mode Shapes}$$

Eigenvalues
or *Natural Frequencies*

Writing this equation for two modes *j* and *k*, (for example the 1st and 3rd mode):

$$\omega_j^2[m]\{u_j\}=[k]\{u_j\} \tag{2-1}$$

$$\omega_k^2[m]\{u_k\}=[k]\{u_k\} \tag{2-2}$$

Now, transpose equation (2-1), and <u>post</u>multiply by $\{u_k\}$

$$\left(\omega_j^2[m]\{u_j\}\right)^T\{u_k\}=\left([k]\{u_k\}\right)^T\{u_k\}$$

Then, because of the “Reversal Law”, $\left(\{[a][b]\}^T=[b]^T[a]^T\right)$, then this is also equal to:

$$\omega_j^2\{u_j\}^T[m]^T\{u_k\}=\{u_j\}^T[k]^T\{u_k\} \tag{2-3}$$

Matrices $[m]$ and $[k]$ are symmetric and so $[m]^T = [m]$ and $[k]^T = [k]$. If we then premultiply equation (2-2) by $\{u_j\}^T$:

$$\omega_k^2 \{u_j\}^T [m]\{u_k\} = \{u_j\}^T [k]\{u_k\} \tag{2-4}$$

We notice now that the right hand sides of equations (2-3) and (2-4) are equal and therefore subtracting equation (2-4) from (2-3) yields:

$$\left(\omega_j^2 - \omega_k^2\right)\{u_j\}^T [m]\{u_k\} = 0$$

Since $\omega_j \neq \omega_k$, then $\underline{\{u_j\}^T [m]\{u_k\} = 0}$ for $j \neq k$ (2-5)

This is the *Orthogonality Condition* for mode shapes $\{u_j\}$ and $\{u_k\}$ including the mass matrix. Then examining equation (2-4) using the orthogonality condition that results from equation (2-5), we see that:

$$\underline{\{u_j\}^T [k]\{u_k\} = 0} \text{ for } j \neq k$$

This is the second Orthogonality Condition including the stiffness matrix.

Equation (2-5) when expanded, is of the form:

$$\{u_1 \quad u_2 \quad \ldots \quad u_n\}_j \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & . & 0 \\ 0 & 0 & 0 & m_n \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ . \\ u_n \end{Bmatrix}_k = 0$$

And if one carries out the multiplication, the orthogonality condition involving mass is obtained in the form:

$$\sum_{i=1}^{n} m_i u_{ij} u_{ik} = 0 \quad \text{for } j \neq k \tag{2-6}$$

Multiplying Equation (2-6) by the natural frequency ω_j^2 and realize that $\omega_j^2 m_i u_{ij}$ is the inertia force associated with mode *j* and hence $\omega_j^2 m_i u_{ij} u_{ik}$ is *force x displacement* or work. Then, equation (2-6) suggests that the total work done by inertia forces of one mode on displacements of any other mode vanishes.

In further considerations we will denote the modal displacements by ϕ_{ij} and all modes listed as columns of a square matrix $[\phi]$,

$$[\phi]=\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & . & \phi_{1n} \\ \phi_{21} & \phi_{22} & \phi_{23} & . & \phi_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ \phi_{n1} & \phi_{n2} & \phi_{n3} & . & \phi_{nn} \end{bmatrix}$$

mode: 1st 2nd 3rd … nth

The modes are *Orthogonal* or *Independent*. We can examine some standard trigonometric functions and their integrals for an analogy to the Orthogonality Condition.

$$\left.\begin{matrix} \int \cos jx \cos kx dx \\ \int \sin jx \sin kx dx \\ \int \sin jx \cos kx dx \end{matrix}\right\} \text{Integrals involving products of harmonic functions.}$$

The trigonometric identities of sums are:

a) $\cos(x+y)=\cos x\cos y-\sin x\sin y$
b) $\cos(x-y)=\cos x\cos y+\sin x\sin y$

Adding a) and b) we obtain:

$\cos x\cos y=\frac{1}{2}\left(\cos(x-y)+\cos(x+y)\right)$, so for $x = y$,

$$\cos jx\cos kx=\tfrac{1}{2}\left(\cos(j-k)x+\cos(j+k)x\right)$$

and for $j \neq k$ (analogous to <u>different</u> modes):

$$\int_0^{2\pi}\cos jx\cos kx dx=\frac{1}{2}\left[\frac{1}{(j-k)}\sin(j-k)x+\frac{1}{j+k}\sin(j+k)x\right]_0^{2\pi}=0$$

or, orthogonal.

and for $j = k$ (analogous to <u>the same</u> mode):

$$\int_0^{2\pi}\cos^2 jx dx=\int_0^{2\pi}\tfrac{1}{2}(1+\cos 2jx dx\neq 0 \qquad \text{or, finite}$$

Generalization of Orthogonality Conditions

It was found that between two different modes, where $j \neq k$ and $\omega_j \neq \omega_k$ that the Orthogonality Condition is:

$$\{\phi_j\}^T [m] \{\phi_k\} = 0 \quad \text{or} \quad \sum_i m_i \phi_{ij} \phi_{ik} = 0 \tag{2-7}$$

Now for $j = k$, $\sum_i m_i {\phi_{ij}}^2 \neq 0$, because $m_i > 0$ and ${\phi_{ij}}^2 > 0$. Thus:

$\{\phi_j\}^T [m] \{\phi_j\} = \sum_i m_i {\phi_{ij}}^2 = M_j^*$, the generalized mass of the j th mode.

The Generalized Mass, recall, is the equivalent "mass" of mode *j* if treated as a single-degree-of-freedom system. More generally, for all modes:

$[\phi]^T [m] [\phi] = [\mathrm{M}^*]$, the diagonal matrix of *Generalized Masses* (2-8)

This can be verified by re-writing $[\phi]$ in terms of partitioned matrices and treating the sub-matrices that are created by this partitioning as elements if they are conformable, as follows:

$[\phi]^T [m] [\phi]$ symbolizes the following triple matrix product:

$$\begin{array}{r} \text{1st mode} \\ \text{2nd mode} \\ \\ \\ \text{nth mode} \end{array} \underset{n \times 1}{\begin{Bmatrix} \{\phi_1\}^T \\ \hline \{\phi_2\}^T \\ \hline . \\ . \\ \{\phi_n\}^T \end{Bmatrix}} \underset{1 \times 1}{[m]} \underset{1 \times n}{\left[\{\phi_1\} \mid \{\phi_2\} \mid . \quad . \quad \{\phi_n\} \right]}$$

and is conformable, i.e.: *n* x *n*

$$\underset{n \times 1}{\begin{Bmatrix} \{\phi_1\}^T [m] \\ \hline \{\phi_2\}^T [m] \\ \hline . \\ . \\ \{\phi_n\}^T [m] \end{Bmatrix}} \underset{1 \times n}{\left[\{\phi_1\} \mid \{\phi_2\} \mid . \quad . \quad \{\phi_n\} \right]} = \begin{bmatrix} M_1^* & 0 & 0 & 0 & 0 \\ 0 & M_2^* & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & M_n^* \end{bmatrix}$$

The second Orthogonality Condition involves the stiffness matrix:

$$\{\phi_j\}^T[k]\{\phi_k\} = 0, \quad \text{when } j \neq k\text{, or} \quad \sum_i k_i \phi_{ij}\phi_{ik} = 0$$

(2-9)

$$\{\phi_j\}^T[k]\{\phi_j\} = K_j^*, \quad \text{when } j = k\text{, or} \sum_i k_i \phi_{ij}{}^2 \neq 0$$

Where K_j^* is the *Generalized Stiffness* (a 1 x 1 matrix or a scalar quantity). Using Equation (2-9), a relation can be derived involving all modes, written as columns in $[\phi]$:

$$[\phi]^T[k][\phi] = [K^*] = [\omega_j^2][M^*] \tag{2-10}$$

and if we look at equation (2-4), then $[K^*]$ is the Generalized Stiffness Matrix; ω_j is the j^{th} natural frequency. To prove equation (2-10), $[\phi]^T$ and $[\phi]$ can be partitioned according to the modes and then the matrices multiplied:

$$[\phi]^T[k][\phi] = \begin{Bmatrix} \{\phi_1\}^T \\ \hline \{\phi_2\}^T \\ \hline . \\ . \\ \{\phi_n\}^T \end{Bmatrix} [k]\left[\{\phi_1\} \mid \{\phi_2\} \mid . \quad . \quad \{\phi_n\}\right]$$

n x 1 1 x 1 1 x n

$$\begin{Bmatrix} \{\phi_1\}^T[k] \\ \hline \{\phi_2\}^T[k] \\ \hline . \\ . \\ \{\phi_n\}^T[k] \end{Bmatrix}\left[\{\phi_1\} \mid \{\phi_2\} \mid . \quad . \quad \{\phi_n\}\right] = \begin{bmatrix} K_1^* & 0 & 0 & 0 & 0 \\ 0 & K_2^* & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & K_n^* \end{bmatrix} = [K^*] = [\omega_j^2][M^*]$$

n x 1 1 x n

With respect to equation (2-3), written for $j = k$:

$$K_j^* = \{\phi_j\}^T[k]\{\phi_j\} = \omega_j^2\{\phi_j\}^T[m]\{\phi_j\} = \omega_j^2 M_j^*$$