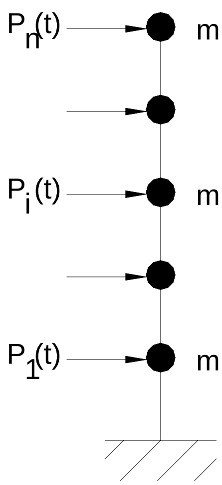
Chapter 3 - Forced Vibration of Multi-Degree-of-Freedom Systems

We will be using modal analysis to solve problems involving Forced Vibration of Multi Degree of Freedom Systems, so the “direct” method which follows is not generally used. This is because the damping term adds a phase shift which generally makes this method impractical for real situations where damping is present.

Forced Undamped Vibration



Equations of motion due to external excitation are readily obtained from the equations of free vibration by adding excitation terms $P_i(t)$ to the right hand side of the MDF equation:

$$m_i\ddot{u}_i + \sum_r k_{ir}u_r = P_i(t) \qquad (3.1)$$

where i = mass 1,2,3 …. n and k_{ir} is the term in the stiffness matrix associated with the force at node r, generated by a unit displacement at node i.

In matrix notation, the equation of motion is:

$$[m]\{\ddot{u}\} + [k]\{u\} = \{P\} \qquad (3.2)$$

We assume that the external forces are “harmonic”, or of the form $P_i(t) = P_i \sin\omega t$ which, in matrix notation is $\{P\} = \{P_o\}\sin\omega t$.

The Particular Solution provides the Steady Response and is $\{u(t)\} = \{u\}\sin\omega t$, where $\{u\}$ is an amplitude vector, describing the amplitude of the individual displacements. This acceleration is of the following form:

$$\{\ddot{u}(t)\} = -\{u\}\omega^2 \sin\omega t \qquad (3.3)$$

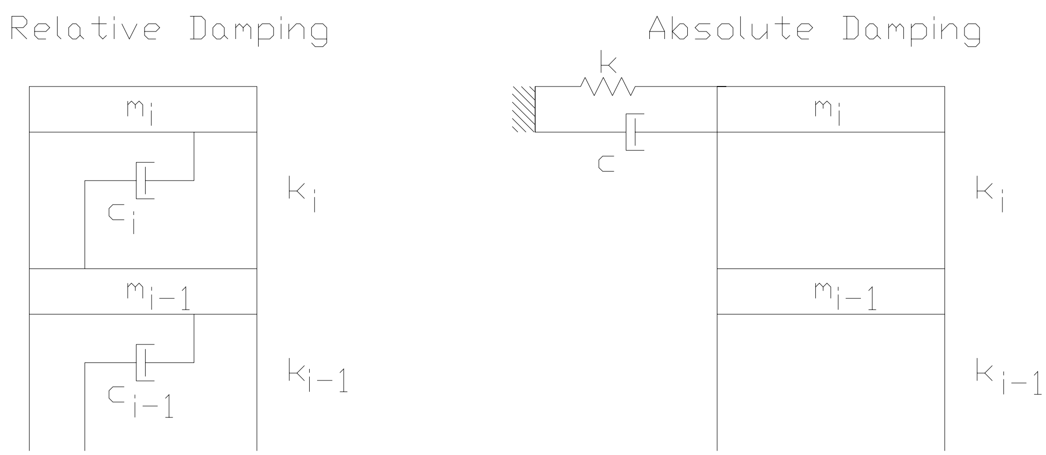
Substituting these expressions for displacement and acceleration in the equation of motion yields:

$$-[m]\omega^2\{u\}\sin\omega t + [k]\{u\}\sin\omega t = \{P_o\}\sin\omega t$$
$$\text{or } \left([k] - \omega^2[m]\right)\{u\} = \{P_o\} \qquad (3.4)$$

This is a set of nonhomogeneous algebraic equations for the unknown amplitudes, $\{u\}$. The frequency, ω is given, since it is the frequency of excitation, and so the resulting amplitudes of vibration can be calculated directly as a solution of simultaneous linear equations, using standard software. A system with n-degrees of freedom has n resonances. At resonances with the natural frequencies, $\omega = \omega_j$, the amplitudes grow to infinite amplitudes when there is no damping present.

Forced Damped Vibration

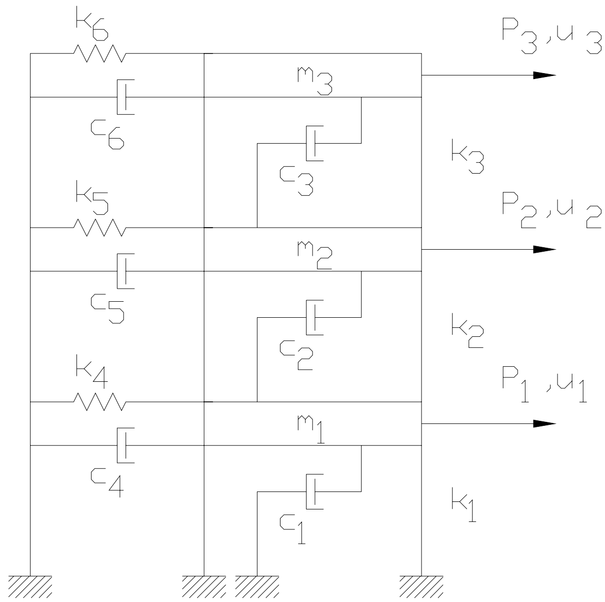
In general, there are two types of damping that one has to examine with damped vibrations.

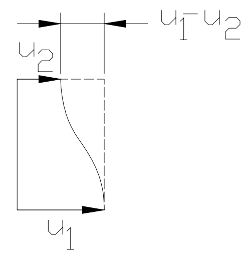


RELATIVE DAMPING	ABSOLUTE DAMPING
- Good for structural damping	- Good for aerodynamic damping
- Depends only on inter-storey motion	- Depends only on storey motion
- Damping force = c_i x relative velocity	- Damping force = c_i x absolute velocity

The relative velocity is the velocity at station (*i*) - the velocity at station (*i*-1)

Since absolute damping depends only on the absolute motion of each mass, the damping matrix is diagonal. Since relative damping depends on the inter-storey motion, then off-diagonal terms are present in the damping matrix. Similarly, stiffness can also be relative, or absolute. In our previous example of the shear building we had relative stiffness in the inter-storey columns





c_2, c_3 - Relative Damping
c_1, c_4, c_5, c_6 - Absolute Damping

k_2, k_3 - Relative Stiffness
k_1, k_4, k_5, k_6 - Absolute Stiffness

We can expand the 3-storey shear building of last week to include both types of stiffness and damping components. Recall that the stiffness constant for the columns of the shear building was of the form:

$$k_i = \frac{12EI_i}{\ell^3} \cdot N \text{ ; where } N \text{ was the number of columns per storey} \qquad (3.5)$$

Applying Newton's second law to the individual masses (*mass x acceleration = sum of forces)*, the equations of equilibrium are:

For the first mass:

$$m_1\ddot{u}_1 = -(k_1 + k_4 + k_2)u_1 + k_2u_2 - (c_1 + c_2 + c_4)\dot{u}_1 + c_2\dot{u}_2 + P_1 \qquad (3.6a,b,c)$$

For the second mass:

$$m_2\ddot{u}_2 = k_2u_1 - (k_2 + k_3 + k_5)u_2 + k_3u_3 + c_2\dot{u}_1 - (c_2 + c_3 + c_5)\dot{u}_2 + c_3\dot{u}_3 + P_2$$

For the third mass:

$$m_3\ddot{u}_3 = k_3u_2 - (k_3 + k_6)u_3 + c_3\dot{u}_2 - (c_3 + c_6)\dot{u}_3 + P_3$$

We can use the double subscripted notation as before to further generalize the equations for each mass:

$$m_i\ddot{u}_i + \sum_{r=1}^{n} k_{ir}u_r + \sum_{r=1}^{n} c_{ir}\dot{u}_r = P_i(t) \quad \text{for each of } i\text{=1,2,…n} \tag{3.7}$$

This in matrix form is:

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = \{P\} \tag{3.8}$$

where:

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},\ \{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix},\ \{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} \tag{3.9}$$

$$[c] = \begin{bmatrix} c_1 + c_2 + c_4 & -c_2 & \\ -c_2 & c_2 + c_3 + c_5 & -c_3 \\ & -c_3 & c_3 + c_6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \tag{3.10}$$

$$[k] = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & 0 \\ -k_2 & k_2 + k_3 + k_5 & -k_3 \\ 0 & -k_3 & k_3 + k_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \tag{3.11}$$

Note that both the damping and stiffness matrices are symmetric, since $k_{ij} = k_{ji}$ and $c_{ij} = c_{ji}$. The elements of the damping matrix are analogous to those elements of the stiffness matrix. The element c_{ij} is the force required at mass *i* (in the direction of u_i to produce a unit velocity at mass *j*, while the velocities at all other masses are zero. The equations of motion are established by forming the stiffness, damping and mass matrices for the whole structure.