Chapter 4 - Modal Analysis

Modal analysis is a general method for analyzing the response of linear multi-degree-of-freedom systems. It is particularly suitable for systems whose properties are frequency independent. The method describes the response in terms of the modes of free vibration whose orthogonality facilitates the solution. Therefore, the analysis of the free vibration (the solution of the eigenvalue problem) must be completed prior to the calculation of the response to external excitation.

The equations of motion can be written in the general form as:

$$[m]\{\ddot{u}\}+[c]\{\dot{u}\}+[k]\{u\}=\{P\} \qquad (4\text{-}1)$$

in which $[m]$, $[c]$ and $[k]$ are mass, damping and stiffness matrices respectively; $\{u\}$ = the displacement vector and $\{P\}$ = the vector of excitation.

The solution to equation (4-1) describing the response is sought in the form of a sum of responses in individual modes of free vibration, Φ_{ij},

$$u_i(t)=\sum_{j=1}^{n}\Phi_{ij}\eta_j(t), \quad i = 1,2 \ldots n \qquad (4\text{-}2)$$

in which:

- Φ_{ij} are modal coordinates of the j^{th} mode. These modal coordinates are independent of time and can be chosen to an arbitrary scale. [In systems with distributed mass, $u_i(t)$ is replaced by $u(x,t)$ and Φ_{ij} by $\Phi_j(x)$];
- $\eta_j(t)$ are new variables associated with mode j and depending on time. They are called generalized coordinates.

To summarize:

The total response at node "i"

Is equal to:

the sum over all modes of:

{the shape function for node "i" and mode "j" x a modal scaling factor (the generalized coordinate of mode "j")}

Equation (4-2) can be interpreted as Fig. 4.1 indicates.

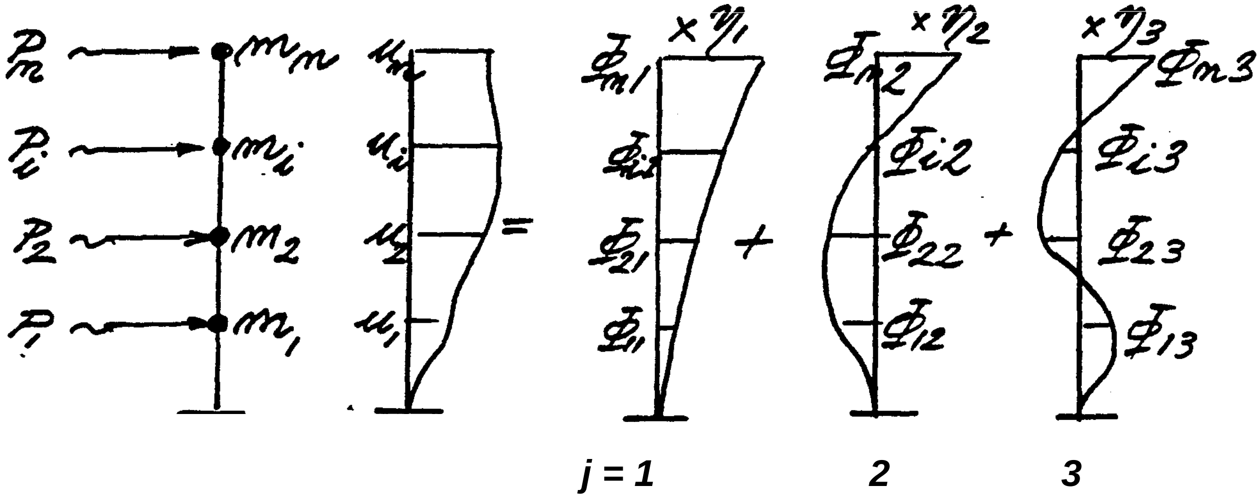


Fig. 4.1 Response in terms of modes.

Equation (4-2) represents a *coordinate transformation* through which one set of n coordinates can be replaced by another set of n independent coordinates. equation (4-2) can be rewritten in matrix form to include all nodes, i = 1, 2..., n:

$$\{u\} = [\Phi]\{\eta\}, \quad \{\dot{u}\} = [\Phi]\{\dot{\eta}\}, \quad \{\ddot{u}\} = [\Phi]\{\ddot{\eta}\} \tag{4-3}$$

where

$$[\Phi] = \begin{array}{c} \left[\begin{array}{c|c|c} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{array}\right] \\ \begin{array}{ccc} j=1 & \quad 2 & \quad 3 \end{array} \end{array} \tag{4-4}$$

Each column in equation (4-4) represents one mode of free vibrations.

Substitute equation (4-3) into equation (4-1) and premultipiy by the transpose of $[\Phi]$ which is $[\Phi]^T$; (this is a matrix in which modes are presented in rows)

$$\underbrace{[\Phi]^T[m][\Phi]}_{[M^*]}\{\ddot{\eta}\} + \overbrace{[\Phi]^T[c][\Phi]}^{\text{a square matrix}}\{\dot{\eta}\} + \underbrace{[\Phi]^T[k][\Phi]}_{[K^*]}\{\eta\} = [\Phi]^T\{P(t)\} \tag{4-5}$$

Equation (4-5) considerably simplifies due to the generalized orthogonality conditions, discussed previously, according to which

$$[\Phi]^T[m][\Phi] = [M^*] \qquad (4\text{-}6a)$$

$$[\Phi]^T[k][\Phi] = [K^*] = [\omega]^2[M^*] \qquad (4\text{-}6b)$$

Hence, the two triple products result in two <u>diagonal</u> matrices which is very advantageous because $[M^*]\{\ddot{\eta}\}$ and $[K^*]\{\eta\}$ are column matrices. Therefore, each equation (4-5), written in ordinary algebraic form, contains only one variable and its second derivative, $\ddot{\eta}_j$.

If it were not for the presence of damping, the equations would be uncoupled, however, damping couples these equations. Clearly, it would be most desirable if the triple matrix product:

$$[\Phi]^T[c][\Phi] \qquad (4\text{-}7)$$

containing the damping constants of the system, resulted in a diagonal matrix because only then may each equation (line) contain only one derivative $\dot{\eta}_j$. In such a case, equation (4-5) represents a set of n independent equations for η_j, $j = 1,2\ldots,n$, that are "uncoupled".

Since $[\Phi]^T$ and $[\Phi]$, two multipliers, are the same in equation (4-7) and equation (4-6), the triple matrix product, equation (4-7), can result in a diagonal matrix only when the damping matrix, [*c*], is proportional to either the mass matrix [*m*] or the stiffness matrix [*k*]. In the first case, [*c*] has to be diagonal and proportional to [*m*],

$$[c] = 2\alpha[m] \qquad (4\text{-}8a)$$

While the second case occurs if

$$[c] = \beta[k] \qquad (4\text{-}8b)$$

The factor 2 in equation (4-8a), is used for convenience, α, β = constants. (Recall that in one degree of freedom the viscous damping constant $c = 2\alpha m$ and $\alpha = D\omega_o$). equation (4-8a) substituted into equation (4-7), yields with respect to equation (4-6a)

$$[\Phi]^T[c][\Phi] = [\Phi]^T 2\alpha[m][\Phi] = 2\alpha[M^*] \qquad (4\text{-}9a)$$

And equation (4-8b) gives

$$[\Phi]^T[c][\Phi] = [\Phi]^T \beta[k][\Phi] = \beta[\Phi]^T\left[\omega_j^2\right][m][\Phi] = \beta\left[\omega_j^2\right][M^*] \qquad (4\text{-}9b)$$

Equation (4-8a) implies that only absolute dampers, may be present, every damping constant, c_i, is proportional to the mass m_i and, finally, the proportionality constant α is the same for all dampers.

Equations (4-6a) and (4-9a) substituted into equation (4-5) give

$$[M^*]\{\ddot{\eta}\} + 2\alpha[M^*]\{\dot{\eta}\} + \underbrace{\left[\omega_j^2\right][M^*]}_{[K^*]}\{\eta\} = [\Phi]^T\{P(t)\} \qquad (4\text{-}10)$$

This is a set of n independent (uncoupled) equations. Each of them has the form

$$M_j\ddot{\eta}_j + 2\alpha M_j\dot{\eta}_j + \omega_j^2 M_j\eta_j = \{\Phi_j\}^T\{P(t)\} \qquad (4\text{-}11)$$

or

$$\ddot{\eta}_j(t) + 2\alpha\dot{\eta}_j(t) + \omega_j^2\eta_j(t) = \frac{p_j(t)}{M_j}, \quad j = 1,2,3 \ldots n \qquad (4\text{-}12)$$

This is identical to the single degree of freedom equation, in which

$$p_j(t) = \{\Phi_j\}^T\{P(t)\} = \sum_{i=1}^{n}\Phi_{ij}P_i(t) \qquad (4\text{-}13)$$

is the generalized force linked to generalized coordinate η_j (i.e the Generalized Force in mode *j*). Equation (4-12) are independent and each of them is exactly equal to the equation of a single degree of freedom system and, therefore, can easily be solved. Thus, whenever one degree of freedom can be solved, many degrees of freedom can be solved too; η_j are obtained from equation (4-12) and substituted into equation (4-2), which was an expression for the motion at mass "*i*":

$$u_i(t) = \sum_{j=1}^{n}\Phi_{ij}\eta_j(t), \; i = 1,2 \ldots n$$

The uncoupled generalized coordinates $\eta_j(t)$ are also called normal coordinates.

The damping constant α occurring in equation (4-12) is calculated for each mode as $\alpha = D_j \omega_j$ in analogy with one degree of freedom. The modal damping ratio, D_j,. can be in some cases calculated, e.g. damping due to soil, fluids or air, in other cases must be estimated.

With the damping matrix proportional to stiffness matrix (equation (4-8b)), uncoupled equation (4-12) is again obtained with 2α replaced by $\beta\omega_j^2$.

The decoupling of equations of motion can be interpreted as a result of the external loads being in effect replaced by a system of fictitious (generalized) loads such that each of them excites just one mode only.

Subscript *i* refers to a component of motion or to mass *i*; subscript *j* identifies the mode.

The approach is general. With the number of masses $\Rightarrow \infty$, a discrete system changes into a distributed one. The only change in modal analysis is that $n = \infty$ and summations in the equations for generalized mass and generalized force change to integration:

$$M_j = \int_0^\ell m(x)\Phi_j^2(x)dx, \text{ the Generalized Mass in mode } j,$$

$$p_j = \int_0^\ell P(x,t)\Phi_j(x)dx, \text{ the Generalized Force in mode } j,$$

Equations (4-12) and (4-2) remain unchanged.

Further solution depends only on the character of the external forces. The principal types of excitation are discussed below.

If the damping is not proportional to either [*k*] or [*m*], the equations of motion can be uncoupled using complex vibration modes. (see Novak, M. and El Hifnawy, L., "Effect of soil-structure interaction on damping of structures," J. of Earthquake Eng. and Structural Dynamics, Vol. 11, 1983, pp. 595-621.)

4.1 HARMONIC EXCITATION

Assume harmonic excitation with frequency was in the case of unbalanced masses of machines, vortex shedding etc. Such forces can be described as:

$$P_i(t) = P_i \cos\omega t \text{ , or } \{P(t)\} = \{P\}\cos\omega t \qquad (4\text{-}14)$$

Where

$$\{P\} = [P_1 \quad P_2 \quad ... \quad P_i \quad ... \quad P_n]^T$$

The generalized forces for mode *j*, are from equation (4-13)

$$p_j(t) = \cos\omega t \sum_{i=1}^{n} \Phi_{ij} P_i \tag{4-15}$$

or

$$p_j(t) = L_j \cos\omega t \,, \qquad L_j = \sum_{i=1}^{n} \Phi_{ij} P_i \,, \tag{4-16}$$

the *Force Participation Factor* in mode *j*

The generalized equation of motion, equation (4-12), is

$$\ddot{\eta}_j(t) + 2\alpha\dot{\eta}_j(t) + \omega_j^2 \eta_j(t) = \frac{p_j(t)}{M_j} = \frac{L_j}{M_j}\cos\omega t \tag{4-17}$$

This is an equation formally identical with that of single degree of freedom systems whose mass is M_j and which is subjected to harmonic excitation with amplitude L_j. This is called the *Force Participation Factor* because its magnitude is a measure of the degree to which the excitation forces participate in the excitation of mode *j*.

The solution of equation (4-17) follows from the SDOF solution found previously,

$$\eta_j(t) = \underbrace{\eta_j \cos(\omega t + \phi_j)}_{\text{Steady State}} + \underbrace{\eta_j^o e^{-\alpha t} \cos(\omega_j t + \phi_j^o)}_{\text{Transient}} \tag{4-18}$$

in which, the first term describes the most important steady state part of the motion (the particular solution). Its amplitude is

$$\eta_j = \frac{L_j}{M_j\omega_j^2} \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_j}\right)^2\right]^2 + 4\left(\frac{\omega}{\omega_j}\right)^2 D_j^2}} = \frac{L_j}{M_j\omega_j^2}\varepsilon_j \tag{4-19}$$

where ε_j = the dynamic magnification factor in one degree of freedom whose natural frequency is ω_j. and damping D_j. and $M_j\omega_j^2 = K_j$ = generalized stiffness. The phase shift of the steady state component

$$\phi_j = -\arctan \frac{2D_j \dfrac{\omega}{\omega_j}}{1-\left(\dfrac{\omega}{\omega_j}\right)^2} \tag{4-20}$$

The transient part of equation (4-18) dies out due to damping; constants η_j^o and ϕ_j^o, if needed, are given by initial conditions.

The real steady motion is from equation (4-2)

$$u_i(t) = \sum_{j=1}^{n} \Phi_{ij}\eta_j = \sum_{j=1}^{n} u_{ij}\cos(\omega t + \phi_j) \tag{4-21}$$

where the amplitude in mode *j* is

$$u_{ij} = \frac{L_j \Phi_{ij}}{M_j \omega_j^2} \frac{1}{\sqrt{\left[1-\left(\dfrac{\omega}{\omega_j}\right)^2\right]^2 + 4\left(\dfrac{\omega}{\omega_j}\right)^2 D_j^2}} = \frac{L_j \Phi_{ij}}{M_j \omega_j^2}\varepsilon_j = \Phi_{ij}\eta_j \tag{4-21a}$$

Thus, the response in each displacement coordinate *i* consists of harmonic components that have the same frequency ω but different amplitudes and phase shifts. At resonance with mode *r*, $\omega = \omega_j = \omega_r$, $\varepsilon_j = 1/2D_r$; the resonant amplitude of the resonating mode and its phase shift are

$$u_{ir} = \frac{L_j \Phi_{ir}}{M_r \omega_r^2} \frac{1}{2D_r}, \qquad \phi_r = -\pi/2 \tag{4-22}$$

With small damping, the resonant amplitude is usually much larger than the non-resonant amplitudes of the other modes and <u>equation (4-22) is sufficient to estimate the resonant amplitudes of the system</u>.

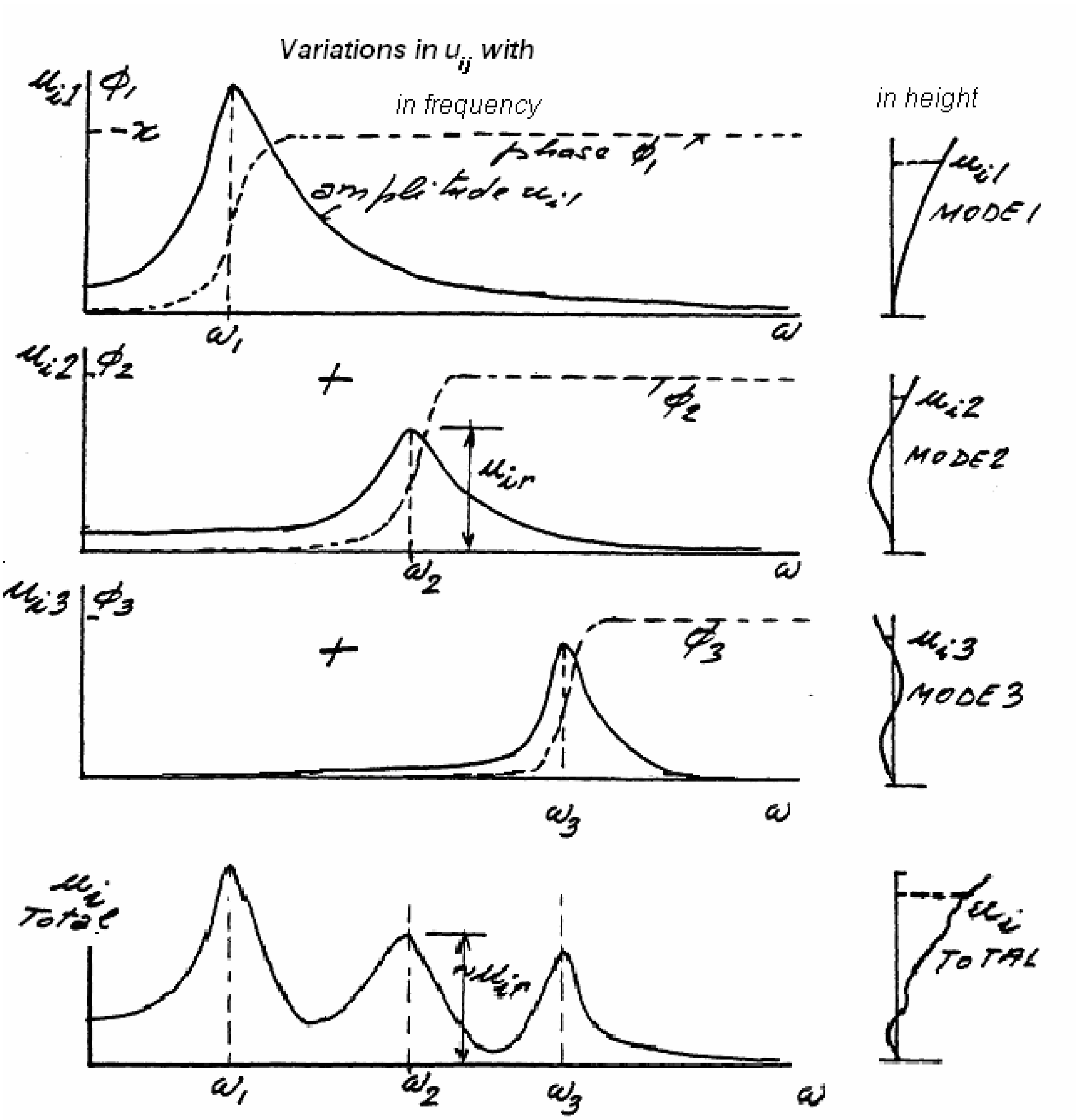
With large damping the contribution from all modes may be more significant. At resonance with the first natural frequency, i.e. for $\omega = \omega_1$, the phase shifts, evaluated from equation (4-20), take on the values ϕ_1=-90° and $\phi_{2,3,.....} \cong 0$. Consequently, the first resonance amplitude $u_i = u_i(\omega_1)$ becomes approximately

$$u_i = \sqrt{u_{i1}^2 + \left(\sum_{j=2}^{n} u_{ij}\right)^2}$$, the total response at mass "*i*", when $\omega = \omega_1$

Similar approximate expressions can be written for the other resonances, *r*, realizing that $\phi_1 \cong -180°$, $\phi_r \cong -90°$ and $\phi_i \cong 0°$ for *i* > *r* (see Fig 4.2).

The superposition of the responses in individual modes is shown in Fig.4.2. Because of the phase differences of the modal components, the resultant amplitude:

$$u_i \leq |u_{i1}| + |u_{i2}| + |u_{i3}| + \ldots\ldots\ldots$$



Fiq. 4.2 Superposition of responses in individual modes.

When using the modal analysis, the coordinates of the modes (eigenvectors) can be chosen to an arbitrary scale. This can be seen from equation (4-21a) and from the application of this formula to one degree of freedom.

In *one degree* of freedom

$$\omega_j^2 = k/m,\ L_r = \Phi P,\ M_r = m\Phi^2,\ \Phi_{ir} = \Phi$$

and the resonant amplitude from equation (4-22).

$$u = \frac{L_r \Phi_{ir}}{M_r \omega_r^2} \cdot \frac{1}{2D_r} = \frac{\Phi P \Phi m}{m\Phi^2 k 2D} = \frac{P}{k}\frac{1}{2D} = u_{st}\frac{1}{2D}$$

as found previously.

Before the modal analysis is started, the modes are sometimes normalized in such a way that $\Phi_{nj} = 1$or $M_j^* = 1$ for each mode. In the latter case, the modes so normalized are called orthonormal modes. Their coordinates are

$$\overline{\Phi}_{ij} = \frac{\Phi_{ij}}{\sqrt{M_j^*}}$$

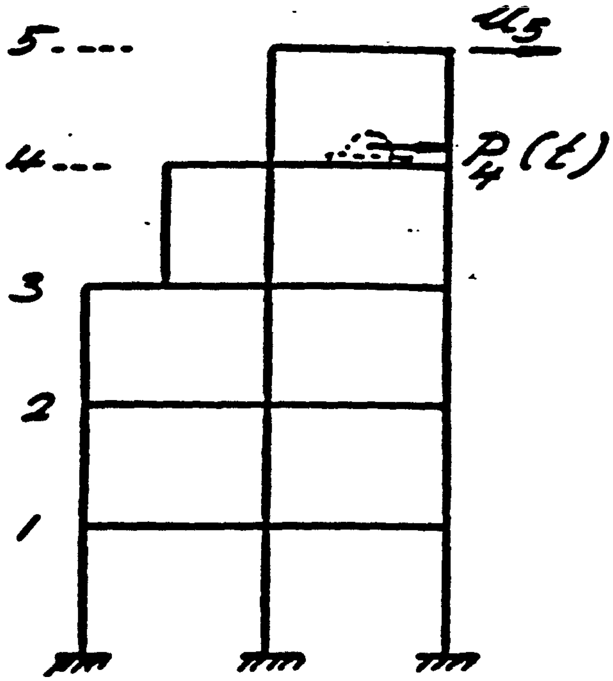
and the generalized mass

$$\overline{M}_j^* = \sum_{i=1}^{n} m_i \Phi_{ij}^2 = 1$$

This normalization is used by some writers but it does not offer any particular advantage.

The modal coordinates generally have the dimensions of the displacements, i.e. translations or rotations; if only translations are involved, they can be taken as dimensionless in which case the generalized coordinates assume the dimension of length and the generalized masses are in kg (or slugs).

Problem 4.1: Consider the five storey shear building analyzed in the tutorial.



The elevator drive produces a harmonic force acting on the 4th floor,

$P_4(t) = P_4 \cos \omega t$
$P_4 = 1.5\ kN$
The frequency ω of the drive is variable.

a) Find the resonant amplitudes of the top (5th) floor at natural frequencies:

$$\omega = \omega_1 \quad , \ \omega = \omega_2 , \ \ \omega = \omega_3$$

and also amplitudes at the operating speed of the elevator motor

$$\omega = \tfrac{1}{2}(\omega_3 + \omega_4)$$

The Damping ratio for all modes is $D_j = 0.01$ and recall that, $\alpha_j = D\omega_j$

b) Evaluate also the physiological effects using Fig. 4.3.

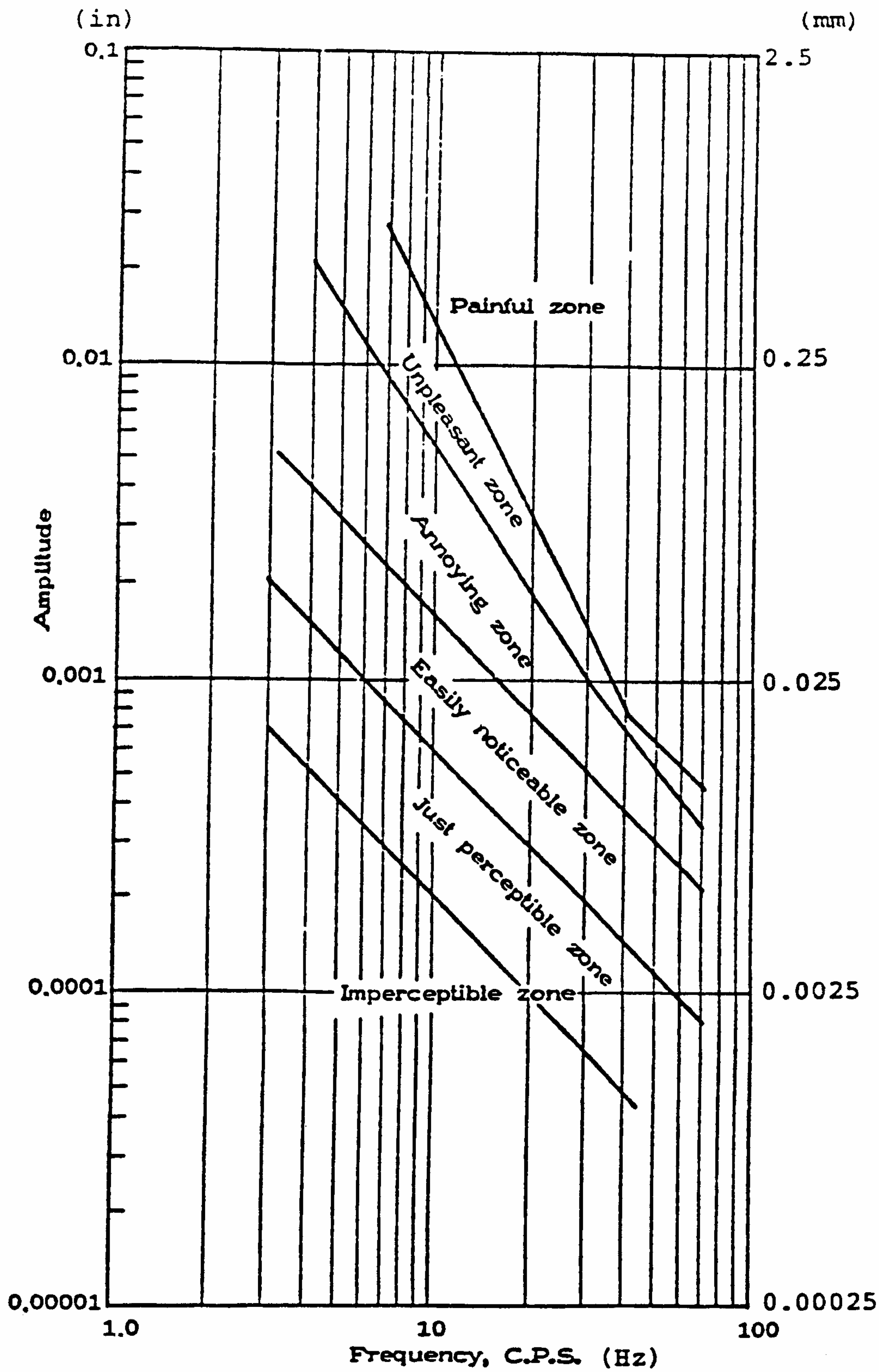


Fig. 4.3 Human Susceptibility to Vibration (after Reiher & Meister)

Static Loading

Static loading can also be conveniently solved by means of modal analysis as a special case of harmonic excitation. The pertinent formulae follow from the preceding paragraph with the frequency of excitation $\omega \Rightarrow 0$ and thus $\cos \omega t = 1$. With $\omega = 0$, loads $P_i(t) = P_i$ and $\phi = 0$. From equation (4-19), the generalized coordinate of mode *j* is:

$$\eta_{i,st} = \frac{L_j}{M_j \omega_j^2} \tag{4-23}$$

and from equation (4-21), the static displacement of mass m_i due to static loads

$$u_{i,st} = \sum_{j=1}^{n} \frac{L_j}{M_j \omega_j^2} \Phi_{ij} \tag{4-24}$$

This is an exact approach, suitable to examine the effect of static loads in statically indeterminate structures if the free vibration modes are known already from previous analysis.

This approach can also be used to find the response to a suddenly applied static load or a rectangular pulse whose duration T_P is sufficiently longer than the natural period of the structure T_1. In these cases, the maximum (peak) response ≤ 2 x the static response.

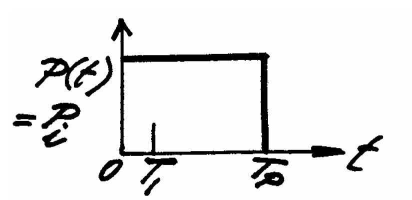


Fig. 4.4

Examples:

4.2 The two storey shear building is exposed to static wind load in the horizontal direction given as P_1 = 30 kN , P_2 = 20 kN. Compute the displacements u_1, u_2 and the stresses in the columns .

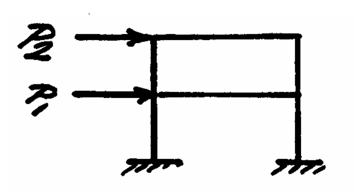


Fig. 4.5

4.3 The five storey shear building is being designed for the Toronto area. Its width is W = 24 m (tributary width for one frame = 6 m). Calculate the wind loads and deflections of the building, the stresses in the lowest columns and maximum acceleration for return period 30 years, exposures A and C, and damping ratio 1%.

4.2 RESPONSE TO GROUND MOTION

Differential equations of motion depend on the nature of damping. The more important case is that of relative damping.

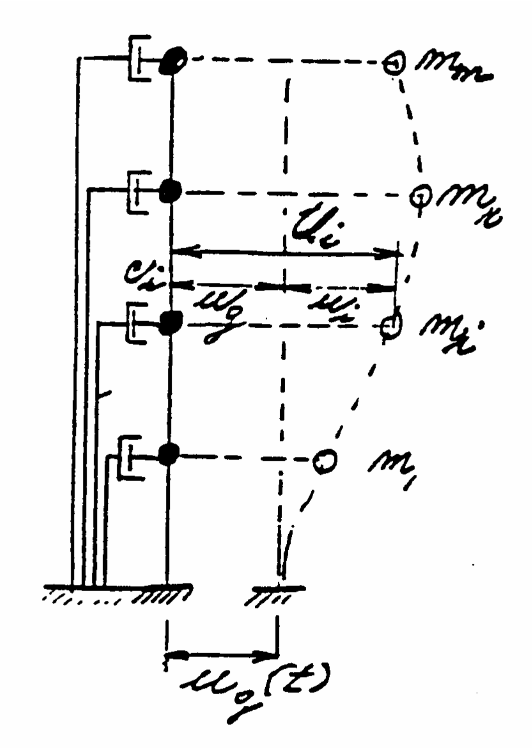


Fig. 4.6

Relative damping. - From Newton's second law, forces acting on mass, m_i are

$$m_i \ddot{U}_i = -\sum_{r=1}^{n} k_{ir}(U_r - u_g) - c_i(\dot{U}_i - \dot{u}_g)$$

where the absolute displacement $U_i = u_g + u_i$ and the relative displacement $u_i = U_i - u_g$. With the damping force at mass m_i described as

$$-c_i(\dot{U}_i - \dot{u}_g) = -2m_i\alpha(\dot{U}_i - \dot{u}_g)$$
$$= -2m_i\alpha\dot{u}_i$$

The differential equations of the motion in terms of relative coordinates u become –

$$m_i\ddot{u}_i + 2m_i\alpha\,\dot{u}_i + \sum_{r=1}^{n} k_{ir}u_r = -m_i\ddot{u}_g\,, \qquad i = 1,2,3,\ \ldots.\ n \qquad \text{(4-25)}$$

This is formally equal to equation (4-1) in which the equivalent exciting forces

$$P_i(t) = -m_i\ddot{u}_g\,, \qquad i = 1,2,3,\ \ldots.\ n$$

The damping satisfies equation (4-8a) and, therefore, modal analysis leads to decoupled equations for generalized coordinates η_i. The generalized force (the sign can be omitted) is:

$$p_j(t) = \sum_i P_i(t)\phi_{ij}$$

or

$$p_j(t) = \ddot{u}_g \sum_{i=1}^{n} m_i \Phi_{ij} = \ddot{u}_g L_j$$

where the earthquake participation factor $L_j = \sum_{i=1}^{n} m_i \Phi_{ij}$

In matrix form, equations (4-25) can be written as

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = -[m]\{1\}\ddot{u}_g \qquad \text{(4-25a)}$$

in which $[c] = 2\alpha[m]$ and $\{1\} = [1 \;\; 1 \;\; 1 \;\; ... \;\; 1]^T$

Equation (4-25a) can also be derived directly realizing that the inertia forces stem from the absolute displacements U_i but the stiffness and damping forces are due to the relative displacements u_i. Then,

$$[m]\{\ddot{U}\} + [c]\{\dot{u}\} + [k]\{u\} = \{0\}$$

Eliminating U, equation (4-25) is obtained.

The relative displacement of mass i is, by equation (4-2),

$$u_i(t) = \sum_{j=1}^{n} \Phi_{ij} \eta_j \qquad \text{(4-25b)}$$

in which the generalized coordinate is given by equation (4-12),

$$\ddot{\eta}_j + 2\alpha\dot{\eta}_j + \omega_j^2 \eta_j = \frac{p_j(t)}{M_j} = \frac{L_j}{M_j}\ddot{u}_g(t) \qquad \text{(4-26)}$$

This is analogous to one degree of freedom in which the solution depends on the type of ground motion.

Transient ground motion produces response whose solution in 1 DOF is given by the Duhamel (convolution) integral:

$$y(t) = \frac{1}{m\omega_o} \int_0^t P(\tau)\, e^{-D\omega_o(t-\tau)} \sin\omega_o(t-\tau) d\tau$$

With $P(\tau) = L_j \ddot{u}_g(\tau)$, substitution gives for the generalized coordinate:

$$\eta_j(t) = \frac{1}{\omega_j}\frac{L_j}{M_j}\int_0^t \ddot{u}_g(\tau)\, e^{-D_j\omega_j(t-\tau)} \sin\omega_j(t-\tau)d\tau$$
$$= \frac{1}{\omega_j}\frac{L_j}{M_j}V_j(t)$$

where

$$V_j(t) = \int_0^t \ddot{u}_g(\tau)\, e^{-D_j\omega_j(t-\tau)} \sin\omega_j(t-\tau)d\tau \qquad (4\text{-}27)$$

The complete response is the sum of responses in individual modes, i.e.,

$$u_i(t) = \sum_{j=1}^{n}\frac{1}{\omega_j}\frac{L_j}{M_j}\Phi_{ij}V_j(t) \qquad (4\text{-}28)$$

Integral $V_j(t) = f(\ddot{u}_g, \omega_j, D_j)$ is exactly the same as in one degree of freedom and can be obtained by numerical integration. *V(t)* has the dimension of velocity, (i.e. m/s). equation (4-28) can be interpreted as Fig. 4.7 indicates.

4.3 SPECTRAL APPROACH TO TRANSIENT MOTION

It is usually not necessary to find the complete time history of the response. The maximum response is decisive in most applications and this is obtained by substituting the maximum value of integral $V_j(t)$ into equation (4-28) using the same notation as in one degree of freedom.

$$V_{\max} = S_v = \text{spectral velocity (pseudo velocity)}$$
$$S_d = \frac{V_{\max}}{\omega_j} = \frac{S_v}{\omega_j} = \frac{S_a}{\omega_j^2} = \text{spectral displacement}$$
$$S_a = \omega_j S_v = \text{spectral acceleration.}$$

Spectral velocities can be computed for any typical earthquake. In Fig. 4.7 , the spectral velocity of El Centro earthquake is given. Smoothed or averaged spectra should be used possibly depending on site conditions (Figs. 4.8 and 4.9). In terms of spectral displacement, the <u>maximum</u> (peak) displacement in the j^{th} mode is:

$$\hat{u}_{ij} = \frac{1}{\omega_j}\frac{L_j}{M_j}\Phi_{ij}S_{V(j)} = \frac{L_j}{M_j}\Phi_{ij}S_{d(j)}$$

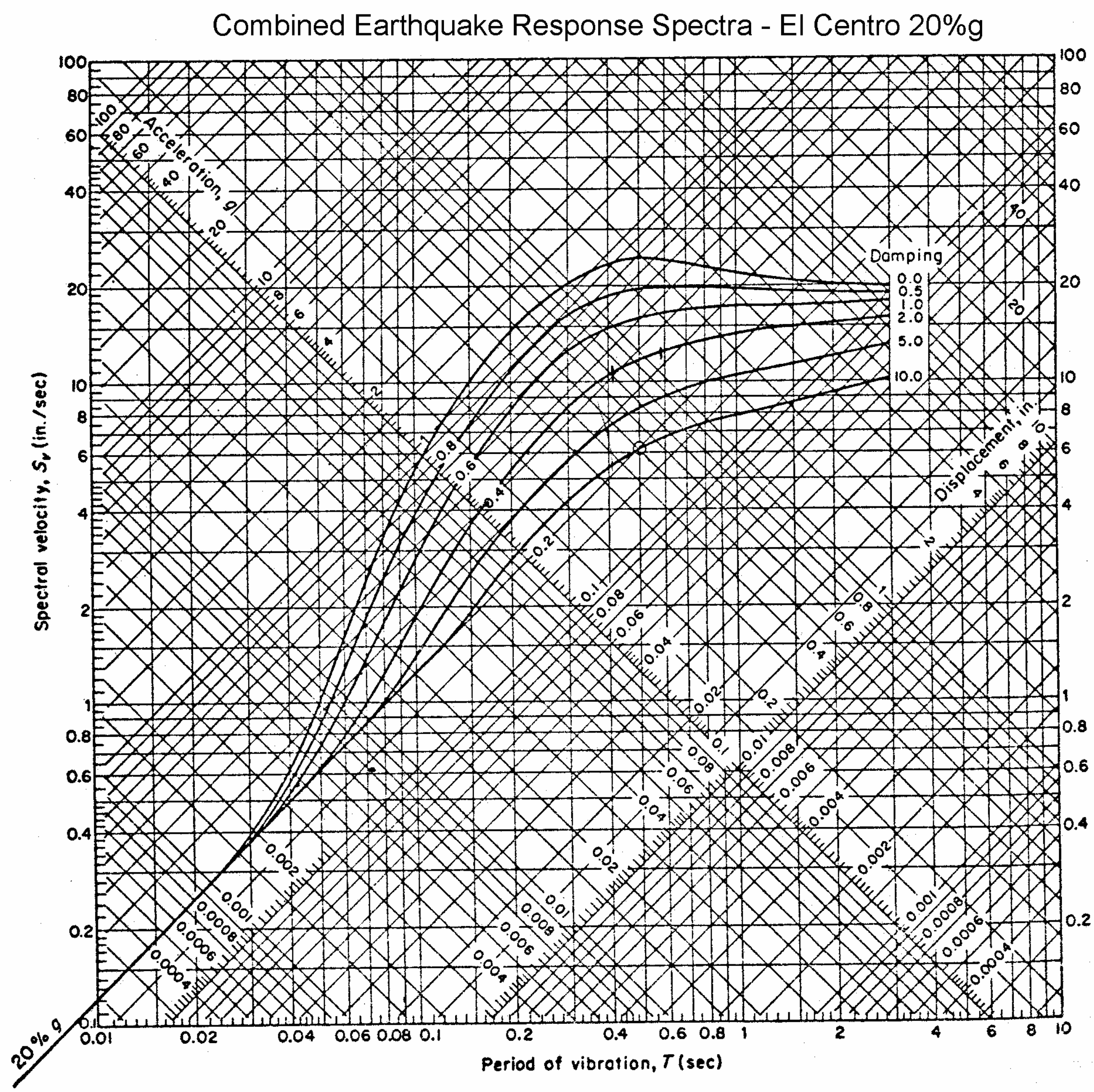


Fig. 4.7 Smoothed El Centro spectra reduced to a maximum acceleration of 20%g

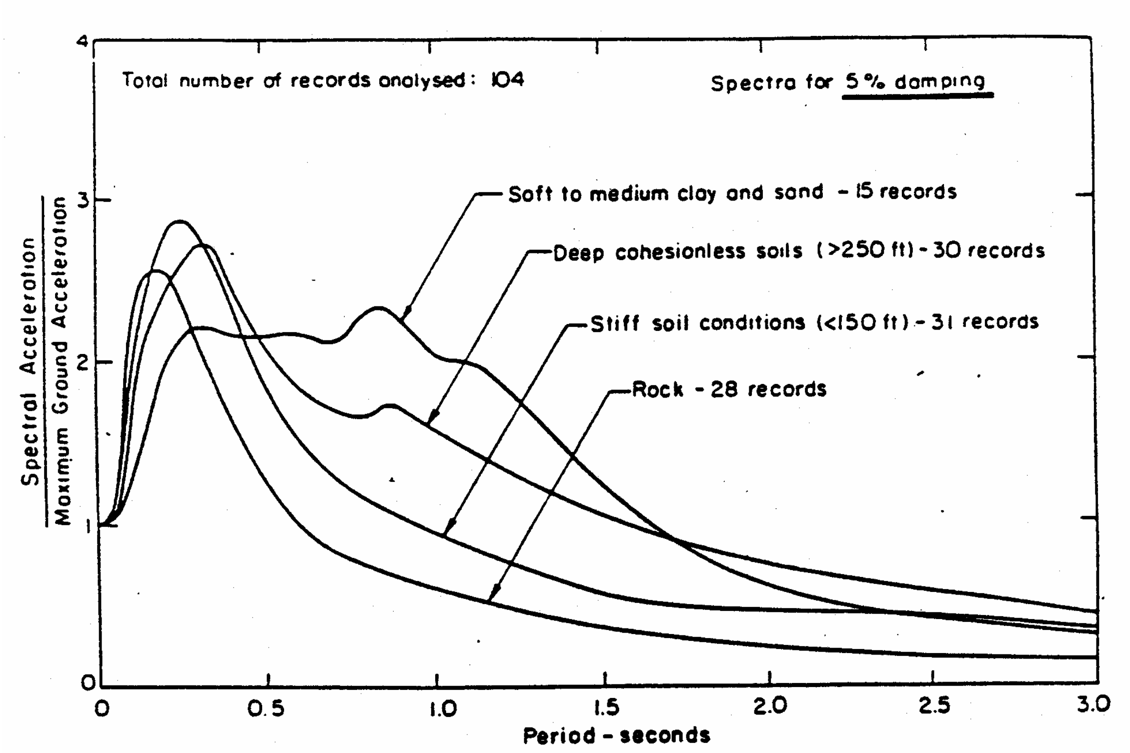


Fig. 4.8 Average acceleration spectra for different site conditions

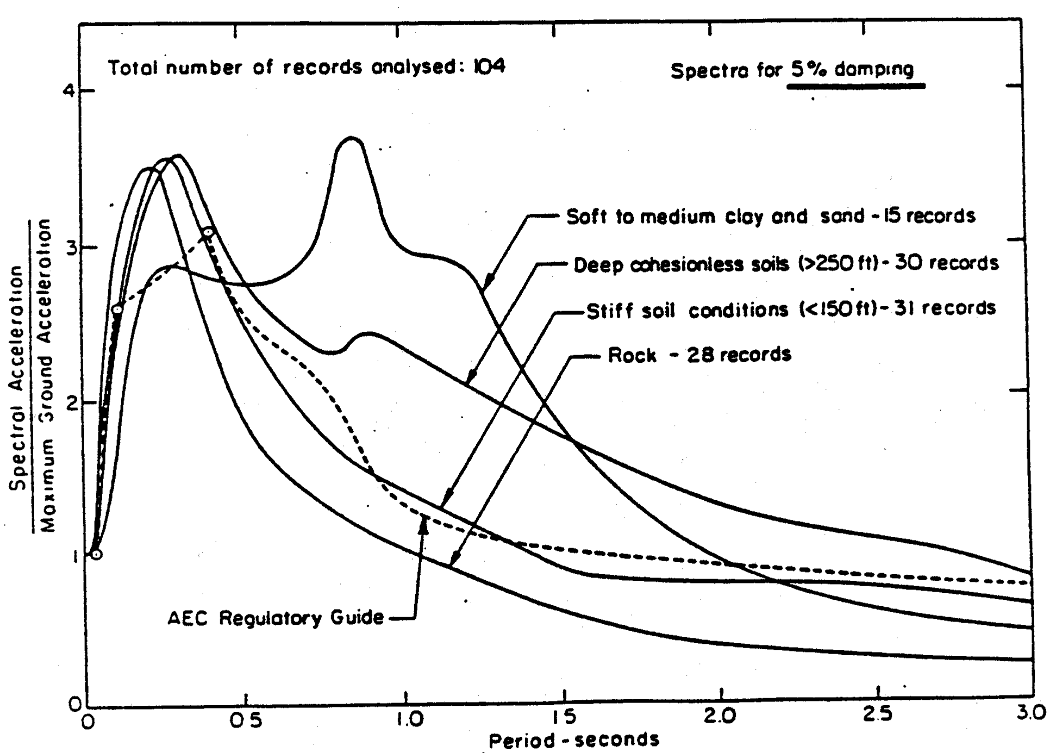


Fig. 4.9 84 percentile acceleration spectra for different site conditions (after Seed, Ugos and Lysmer, 1974)

The effective acceleration for mode *j* is

$$\ddot{u}_{ij} = \omega_j^2 \hat{u}_{ij} = \omega_j^2 \left(\frac{1}{\omega_j} \frac{L_j}{M_j} \Phi_{ij} S_{v(j)} \right) = \frac{L_j}{M_j} \Phi_{ij} S_{a(j)}$$

The stresses can be computed with a static load equal to the maximum effective earthquake force acting on m_j in mode *j* defined as:

$$q_{ij} = m_i \ x \ \text{(accel. Amplitude)} = m_i \omega_j^2 \hat{u}_{ij} = m_i \frac{L_j}{M_j} \Phi_{ij} S_{a(j)}$$

The total maximum base shear which is a measure of earthquake loading is, for mode *j:*

$$Q_j = \sum_{i=1}^{n} q_{ij} = \frac{L_j}{M_j} S_{a(j)} \sum_{i=1}^{n} m_i \Phi_{ij} = \frac{L_j^2}{M_j} S_{a(j)}$$

The total response is a sum of responses in individual vibration modes (Fig. 4.10).

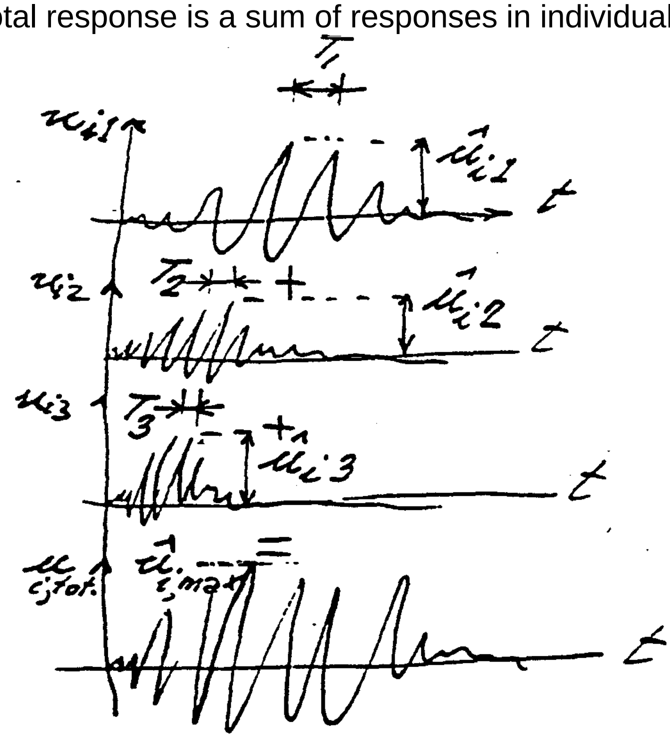


Fig. 4.10

The peak in individual modes does not appear at the same time. Its accurate value could be obtained from the resultant time history. Approximately, the maximum response of mass m_i is:

$$\hat{u}_{i,\max} \cong \sqrt{\hat{u}_{i,1}^2 + \hat{u}_{i,2}^2 + \hat{u}_{i,3}^2 + ...} = \sqrt{\sum_j \hat{u}_{i,j}^2}$$

One can write similar expressions for all types of responses:

e.g. Base Shear: $Q_{\max} = \sqrt{\sum_j Q_j^2}$

Stress at location "*i*" $\sigma_{i,\max} = \sqrt{\sum_j \sigma_{i,j}^2}$

With ***harmonic motion*** of the ground:

$$u_g(t) = u_g \cos \omega t, \;\; \ddot{u}_g(t) = -u_g \omega^2 \cos \omega t$$

and loads:

$$P_i(t) = -m_i \ddot{u}_g(t) = m_i \omega^2 u_g \cos \omega t \; ; \; p_j(t) = \sum_{i=1}^{n} P_i(t) \Phi_{ij} = \omega^2 u_g \cos \omega t \sum_{i=1}^{n} m_i \Phi_{ij}$$

Steady state response is from equations (4-19) and (4-21), denoting $L_j = \sum_{i=1}^{n} m_i \Phi_{ij}$,

$$u_i(t) = u_g \sum_{j=1}^{n} \frac{L_j \omega^2}{M_j \omega_j^2} \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_j}\right)^2\right]^2 + 4\left(\frac{\omega}{\omega_j}\right)^2 D_j^2}} \Phi_{ij} \cos(\omega t + \phi_j)$$

The resonant amplitude in mode *j* is;

$$u_{i(r)} = u_g \frac{L_j}{M_j} \Phi_{ij} \frac{1}{2D_j}$$

4.4 MODAL EQUIVALENT MODEL

In earthquake engineering, the "modal equivalent model" is sometimes used to represent the structure. This model comprises *n* single degree of freedom systems whose damping and natural frequencies are identical to those of individual vibration modes and the masses and their heights are so determined that identical base moments and base shears are obtained from both mode *j* and the equivalent model (Fig.4.11).

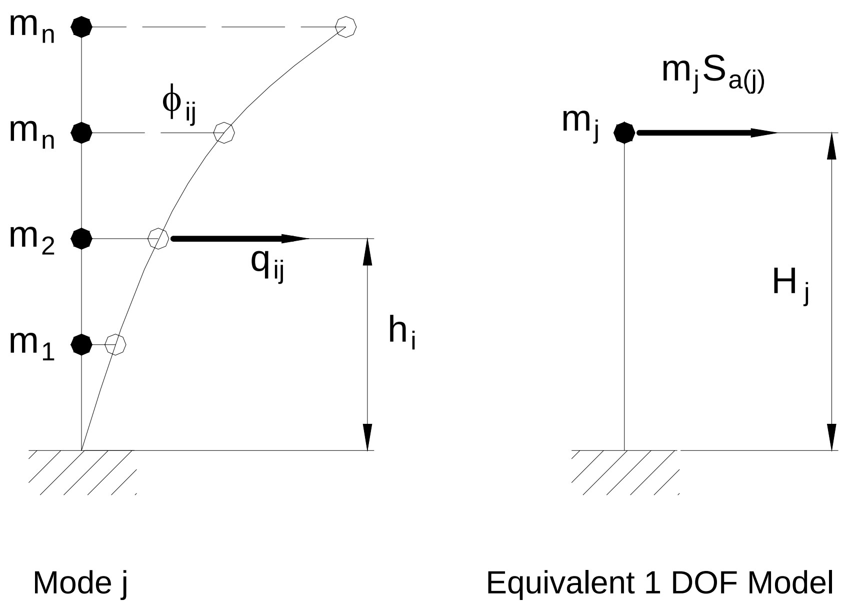


Fig. 4.11 Structural response in mode j and its equivalent 1 DOF model

Thus, for the base shear:

$$Q_j = \frac{L_j^2}{M_j} S_{a(j)} = m_j S_{a(j)}$$

and from here the equivalent mass is:

$$m_j = \frac{L_j^2}{M_j}$$

The equality of base moments (overturning moments) requires:

$$\sum_{i=1}^{n} q_{ij} h_i = m_j S_{a(j)} H_j$$

which yields

$$H_j = \frac{\sum_{i=1}^{n} q_{ij} h_i}{L_j^2 S_{a(j)}} M_j = \frac{\sum_{i=1}^{n} m_i \frac{L_j}{M_j} \Phi_{ij} S_{a(j)} h_i}{L_j^2 S_{a(j)}} M_j$$

After abbreviation the equivalent height is:

$$H_j = \frac{\sum_{i=1}^{n} m_i \Phi_{ij} h_i}{L_j}$$

All the modes can be represented as Fig.4.12 indicates.

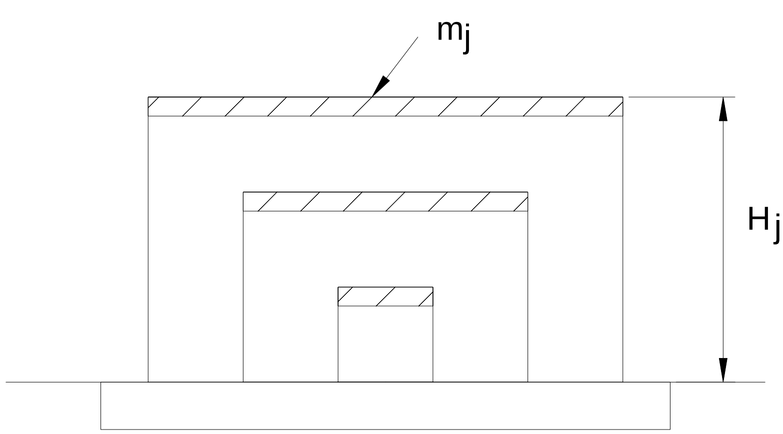


Fig. 4.12 Representation of an *n*-degree of freedom structure by *n* 1 DOF systems

Problem 4.4

a) Calculate the earthquake response, i.e. displacements, earthquake forces, base shear and stresses in the lowest columns of the five storey shear building due to the 20% El Centro. Consider all modes and damping ratio 2%.

b) Calculate the response of the top floor and the base shear of the same five storey building considering the four average acceleration spectra for different site conditions (Fig.4.8), maximum ground acceleration 20% *g*, damping 5% and the first mode only.

Problem 4.5

Calculate the resonant amplitudes of the five storey shear building due to harmonic horizontal ground motion $u_g(t)$, whose amplitude is 0.01 in and frequency ranges from 0 to 1.2ω_5. Assume D_j = 0.01.

Problem 4.6

Adjust the Duhamel integral for numerical integration by taking all factors containing t in front of the integration sign and prepare a flow chart for the computation of the complete time history u_i (t) from equation (4-28).

Answers to Problem 4.4a: (for $\ell_1 = 4.00m$)

Amplitudes:	u_{51} = 4.8x10^{-3} m u_{52} = l.6xl0^{-4} m u_{53} = 2.2x10^{-5} m	f_1 = 5.44 f_2 = 13.90	ω_1 = 34.16
Base shear:	Q1 = 424.82 kN		
Column stress: (first mode)	σ_1 = 36.45 MPa for outside column σ_1' = 42.53 MPa for inside column		