Chapter 5 - Response To Random Loads

5.1 General

A random process differs from the deterministic processes dealt with in the preceding chapters in that it cannot be accurately predicted mathematically even if the past time history is known. Such a process is most meaningfully described in statistical terms. The basic statistical characteristics are reviewed first, presuming that the random process *x(t)* may represent loading of the structure or its response.

A random process may be given in the form of one representative time history (Fig. 5.1) or by a set of sample functions collected into an ensemble (Fig. 5.2). When the statistical characteristics are extracted from the one long time history, the procedure involved is called temporal averaging. The analysis of the ensemble is known as ensemble averaging. Ideally, the time histories should extend from $t \to -\infty$ to $t \to +\infty$

5.2 Basic Statistical Characteristics

Probability density function, $p(x)$. This function defines the probability that *x* will have a value in the range from *x* to *x+dx*. The typical bell-like shape of this function is indicated in Fig. 5.3. Because all values may occur:

$$\int_{-\infty}^{+\infty} p(x)dx = 1 \qquad (5\text{-}1)$$

Probability distribution function, P(x). This function defines the probability of *x* being smaller than or equal to a certain value α and thus:

$$P(x) = \Pr(x \le a) = \int_{-\infty}^{a} p(x)dx \qquad (5\text{-}2)$$

This function is characterized by an S-like shape and is always bounded by the limits $0 < P(x) < 1$ (Fig. 5.4).

Mean value or expected value. This is the mean or average value of the *function* and can be defined for the function *x(t)* as:

$$\bar{x} = \frac{1}{\underset{T \to \infty}{2T}} \int_{-T}^{+T} x(t)dt \qquad (5\text{-}3)$$

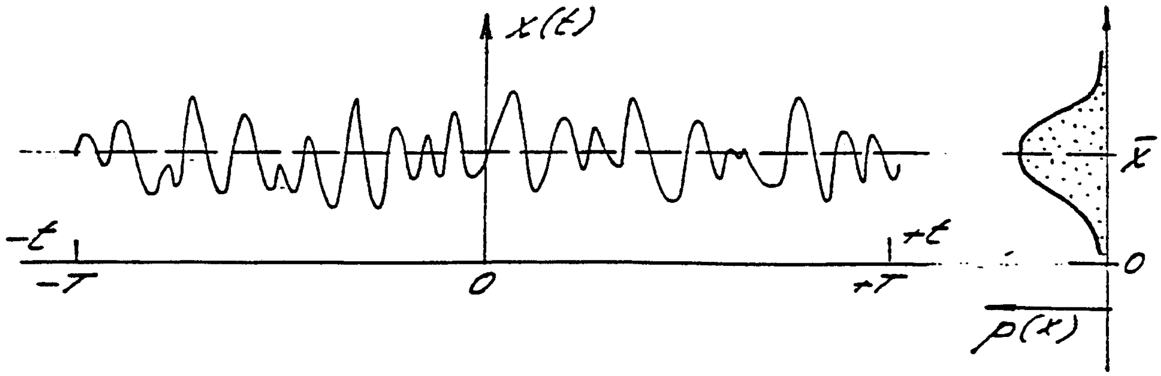


Fig. 5.1 A random function

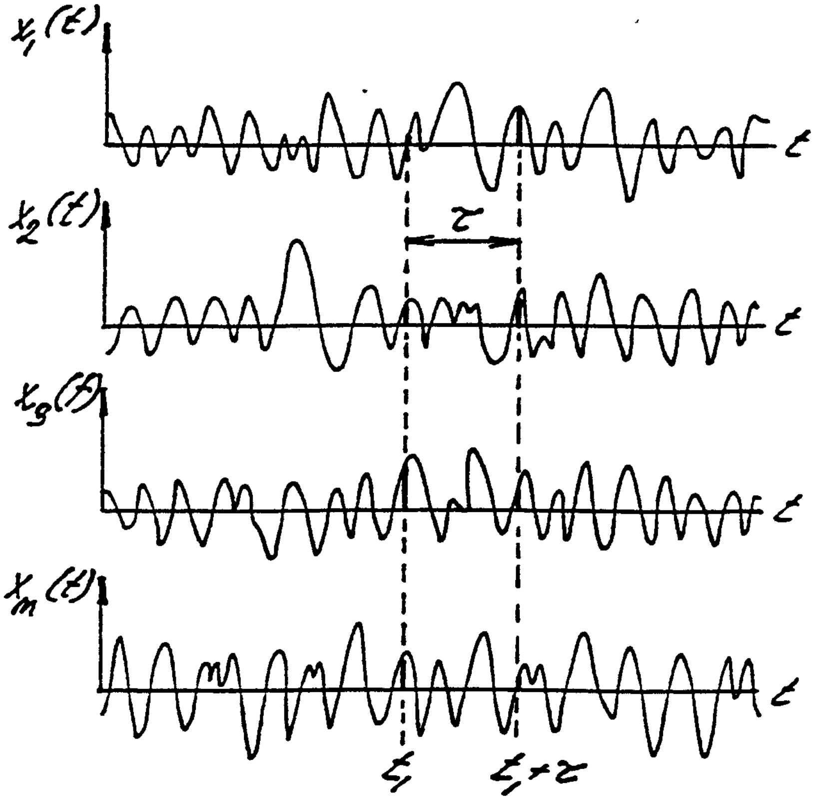


Fig. 5.2 Ensemble of samples of a random function

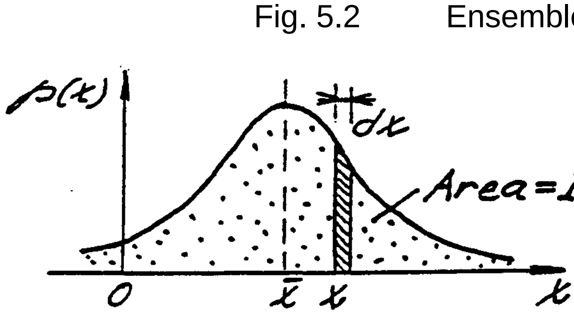


Fig. 5.3 Probability Density

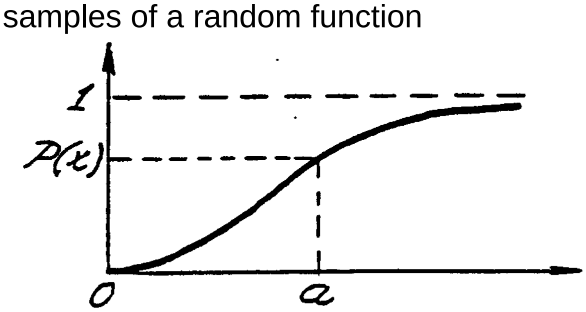


Fig. 5.4 Probability Distribution

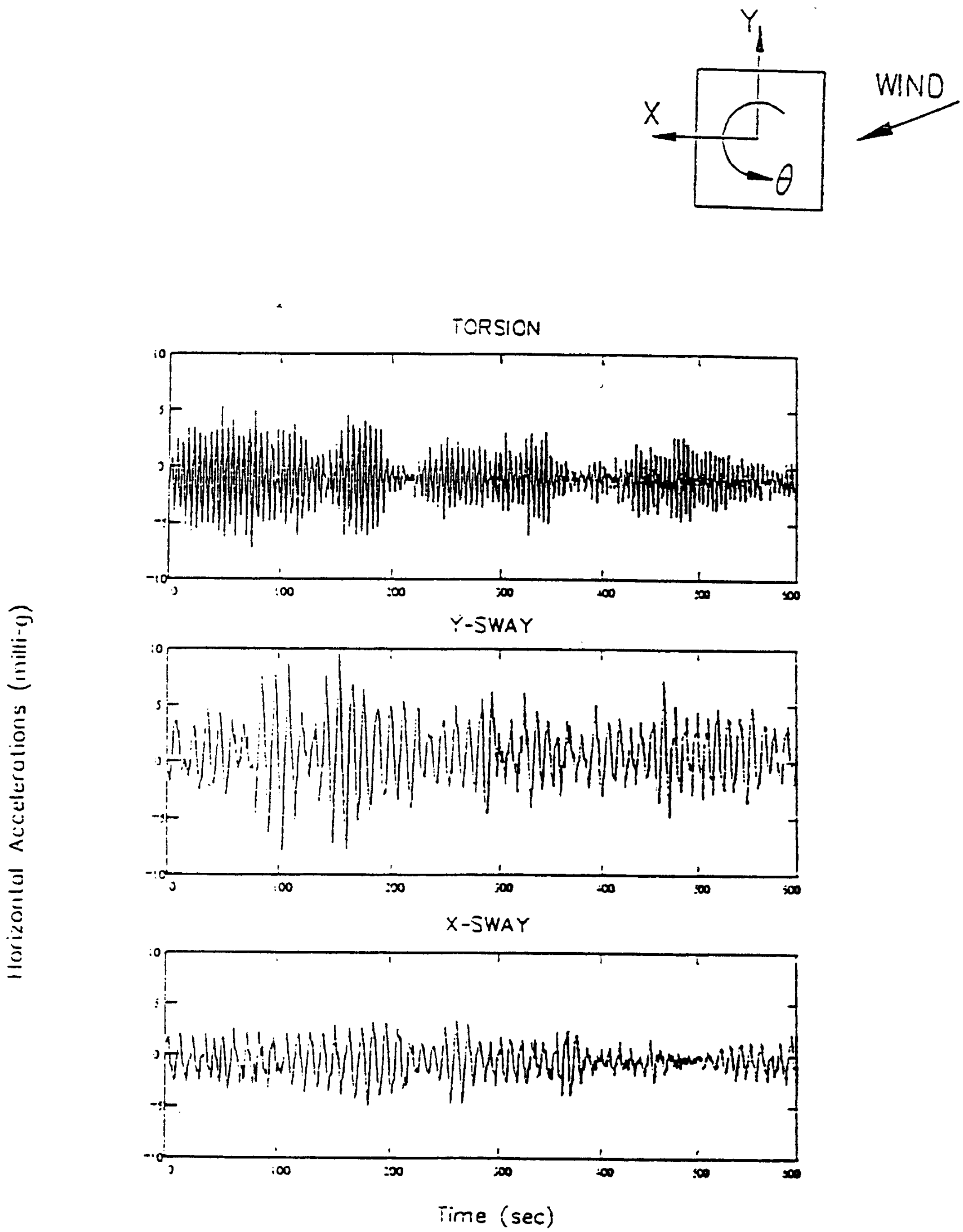
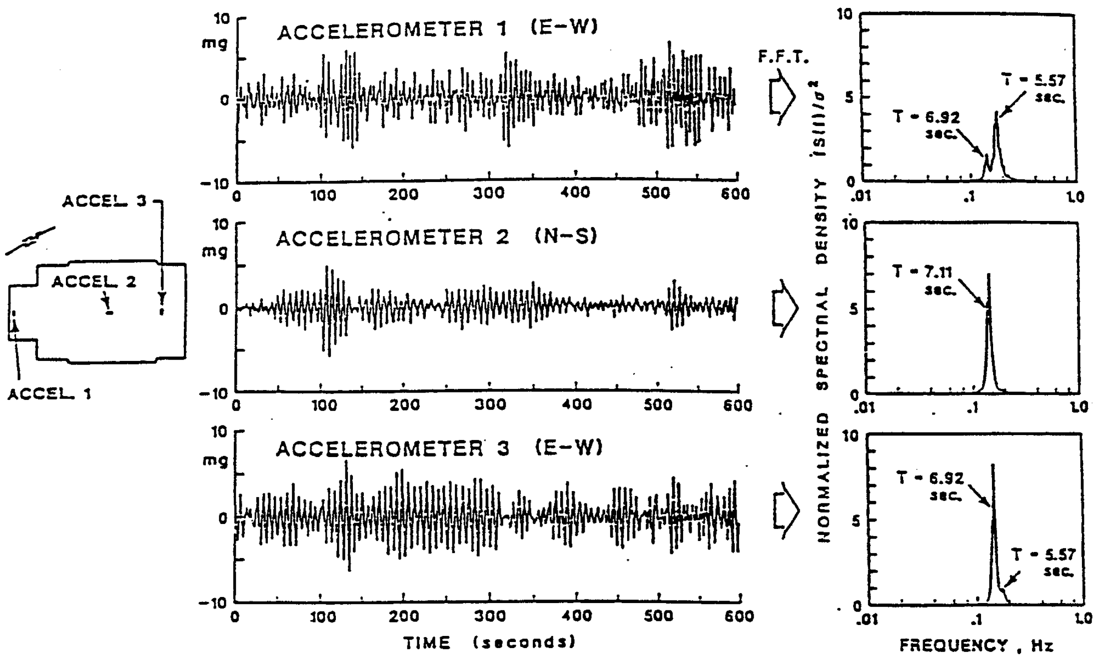


Fig. 5.2a Typical Signatures of wind-Induced Accelerations of a Tall Building



The above components of acceleration can be combined
to estimate the peak resultant acceleration as follows:

$$\hat{a}_R \leq \sqrt{\hat{a}_x^2 + \hat{a}_y^2 + r^2\ddot{\theta}^2}$$

Summary of Fundamental Periods					
Mode	Analytical Estimate (sec)	During Low Winds Avg. (sec)	During Low Winds C.O.V. (%)	During High Winds Avg (sec)	During High Winds C.O.V. (%)
1 N-S	9.0	6.26	5.6	7.16	1.4
2 E-W	7.25	6.32	2.4	6.89	1.9
3 Torsion	5.5	4.75	4.0	5.69	3.1

Fig. 5.2b) Observed Accelerations and Corresponding Power Spectra of a 43-Storey Building as well as Fundamental Periods

Realizing that all values have a total probability of occurring equal to 1, the mean value can also be expressed using the probability density function as

$$\bar{x} = \int_{-\infty}^{+\infty} x p(x) dx \qquad (5\text{-}4)$$

Thus, the value $\bar{x}$ can be viewed as the coordinate of the centroid of the area under the curve *p(x)*.

Finally, the mean value can be obtained from the ensemble of samples by cutting through the ensemble at a certain time, e.g. t_1, and calculating the average value of the samples (Fig. 5.2). This gives

$$\bar{x} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i(t_1) \qquad (5\text{-}5)$$

Other *notations* used are $\bar{x} = E(x) =< x >$.

Mean-square value is the average value of $x^2(t)$, is denoted, as $\overline{x^2} = E(x^2) =< x^2 >$ and follows from the time history of *x(t) as*

$$\overline{x^2} = \underset{T \to \infty}{\frac{1}{2T}} \int_{-T}^{+T} x^2(t) dt \qquad \text{(1}^{\text{st}}\text{ moment about the origin)} \qquad (5\text{-}6)$$

or from the ensemble as

$$\overline{x^2} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} x_i^2(t_1) \qquad (5\text{-}7)$$

Variance is the mean-square value of the difference from the mean. Its notations are $\sigma_x^2 = (x - \bar{x})^2$ and its value is:

$$\sigma_x^2 = \underset{T \to \infty}{\frac{1}{2T}} \int_{-T}^{+T} (x - \bar{x})^2 dt \qquad \text{(2}^{\text{nd}}\text{ moment about origin)} \qquad (5\text{-}8)$$

or from the probability density function:

$$\sigma_x = \int_{-\infty}^{+\infty} (x - \bar{x})^2 p(x) dx \qquad (5\text{-}9)$$

Standard deviation is the square root of variance and is, therefore, also called the root-mean-square value of x, or briefly r.m.s. Standard deviation is usually denoted as:

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\overline{(x-\bar{x})^2}} \tag{5-10}$$

With regard to equations (5-1) and (5-9), the standard deviation can be viewed as the radius of gyration of *p(x)* about $\bar{x}$.

Standard deviation is an important magnitude and together with the mean value $\bar{x}$ are the most important parameters which can characterize a probability distribution function.

Gaussian or Normal probability density and distribution functions are defined as

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}} \tag{5-11}$$

and

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \int_{-\infty}^{x} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}} dx \tag{5-12}$$

This distribution is called “Normal” because it fits most natural phenomena. It can be shown that a force described by a normal distribution produces response of linear systems which also has a normal distribution.

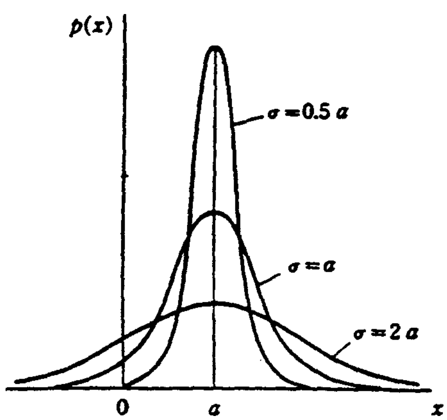


Fig. 5.5 Normal Distributions with $\bar{x} = a$ and different values of σ

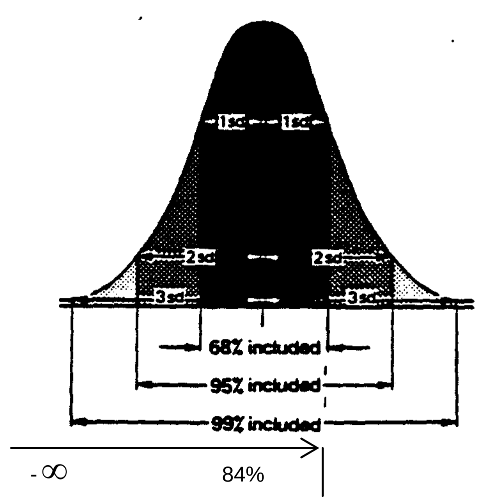


Fig. 5.6 Probabilities associated with multiples of standard deviation $\sigma = sd$

The normal distribution is defined by the eman value, $\bar{x}$, and standard deviation $\sigma_x = \sigma$. The magnitude of σ indicates the spread of the values of *x* about the mean (Fig. 5.5). The dimensionless measure of this spread is the *coefficient of* variation defined as the ratio $\sigma / \bar{x}$.

The normal distribution is widely used and various probabilities following from it are tabulated. For example, the probability of *x* being smaller than or equal to the mean plus one standard deviation is 84%, i.e.

$$P(x \le \bar{x} + \sigma) = 84\%$$

This probability characterizes the pseudovelocity spectra plotted in Fig. 4.9. Other probabilities are indicated in Fig. 5.6.

A given probability density of the process describes the percentage (proportion) of time for which *x* takes on values in a certain range. However, it does not provide any information on the rate of change in *x(t)* i.e., on the frequency characteristics of the process. A more complete description of a random process is contained in further statistical characteristics called correlation functions and power spectral densities.

Autocorrelation function or more briefly correlation function is defined as the mean value of the product of *x(t)* and $x(t+\tau)$ where τ is a time lag, i.e.

$$R_x(\tau) = \overline{x(t)x(t+\tau)} = \underset{T\to\infty}{\frac{1}{2T}} \int_{-T}^{+T} x(t)x(t+\tau)dt \qquad \text{(5-13a)}$$

From the ensemble (Fig. 5.2), the correlation function is obtained as

$$R_x(\tau) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i(t)x_i(t+\tau) \qquad \text{(5-13b)}$$

By evaluating the average products for different values of τ, the correlation function of a process is established. The correlation function is even, droops either in a smooth or oscillatory way and has the following properties:

$$R(0) = \overline{x^2}, \quad R(\infty) = 0, \quad R(\tau) = R(-\tau), \quad \frac{d}{d\tau}R(0) = 0$$

Since τ is real time in seconds, the correlation *function* indicates the speed with which the correlation of the process diminishes and the time within which it vanishes (Fig. 5.7).

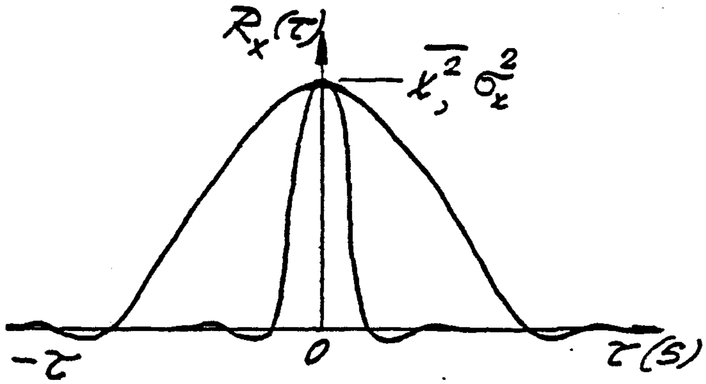


Fig. 5.7 Typical Autocorrelation Functions

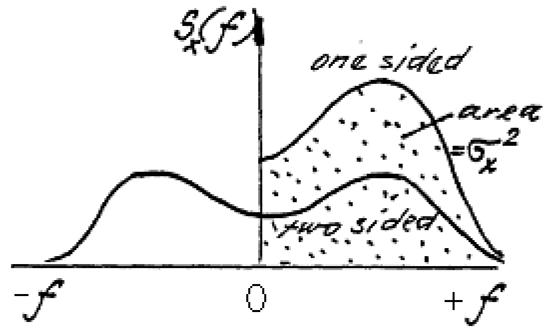


Fig. 5.8 Two-sided and one-sided Power Spectral Densities

The above statistical characteristics allow for a few more definitions.

If the statistical characteristics are independent of the reference time t_1 (Fig. 5.2) the process is called sta*tionary:* if they depend on time t_1 the process is *nonstationary.* The discussion here is limited to stationary processes.

If the process is stationary and the temporal averages are equal to the ensemble averages, the process is *ergodic.*

A centric process is a stationary process with $\bar{x}$ = 0. *Covariance* is one of the measures of the extent to which two random variables x (t) and y (t) are related to each other or correlated. Covariance is defined as

$$\sigma_{xy}^2 = \overline{x(t)y(t)} = < x(t)y(t) > \qquad (5\text{-}14)$$

When x(t) and y(t) are completely independent $\sigma_{xy}^2 = 0$;

$\sigma_{xy}^2 \neq 0$ indicates correlation between the two variables, e.g. input force and response.

In stationary processes the mean value *x* is a constant and it is, therefore, (in favour of numerical accuracy to separate the mean value from the process and analyze just the fluctuating random part of it), $x'(t) = x(t) - \bar{x}$. If *x(t)* is a load, $\bar{x}$ is its static component which produces static deflection about which the structure oscillates due to the effect of the fluctuating component $x'(t)$. Thus, the statistical analysis can be limited to the fluctuating component because $R_x(\tau) = R_{x'}(\tau) + \bar{x}^2$.

Power Spectral Density, S (f). This function describes the energy distribution of the process with regard to frequency and is defined as the Fourier Transform of the autocorrelation function $R(\tau)$, i.e.

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau \qquad (5\text{-}15)$$

This transformation yields an even, two-sided power spectrum indicated in Fig. 5.8. Because negative frequencies do not have a technical meaning, it is usually preferable to define a one-sided power spectrum for positive frequencies only. The area under both types of the spectra has to be the same and thus, the magnitude of the one-sided spectrum is twice the magnitude of the two-sided spectrum (Fig. 5.8). Splitting the integration interval into two, $-\infty$ to 0 and 0 to $+\infty$ and recalling that $R(\tau) = R(-\tau)$, the Fourier transform of the autocorrelation function reduces to a cosine Fourier transform and the one-sided power spectrum becomes

$$S(f) = 4\int_{0}^{\infty} R(\tau) \cos 2\pi f\tau d\tau \qquad (5\text{-}16)$$

The inverse Fourier transform of the spectrum yields the correlation function,

$$R(\tau) = \int_{0}^{\infty} S(f) \cos 2\pi f\tau d\tau \qquad (5\text{-}17)$$

The Fourier transform pair defined by equations (5-16) and (5-17) is also known as the Wiener-Khintchin relationship.

The term power spectrum stems from electrical applications in which it has the following physical meaning. Assume that *x(t)* is random voltage filtered through a narrow band filter and that the power passed by the filter is measured. Then, this power is proportional to the bandwidth of the filter and the spectral density of *x(t)* at the centre frequency of the filter. The total power describes the variance of the signal and thus,

$$\overline{x^2} = \int_{0}^{\infty} S_x(f) df = \int_{0}^{\infty} S_x(\omega) d\omega \qquad (5\text{-}18)$$

Expressing $d\omega = 2\pi df$, equation (5-18) gives

$$S(f) = 2\pi S(\omega) \qquad (5\text{-}19)$$

Equation (5-18) defines the most important property of the spectrum and also suggests the dimension of a spectrum, because $S(f)df$ must have a dimension of x^2. Consequently, $S_X(f)$ is in (dimension of x^2)/frequency. Thus if x is displacement, $S_x(f)$ is in $m^2/s^{-1} = m^2 s$; the power spectrum of acceleration is similarly $m^2/s^4/s^{-1} = m^2/s^{-3}$

Other forms of power spectra used are the normalized spectrum and the logarithmic spectrum.

Normalized spectrum S'(f) is defined as $S'(f) = S(f)/\sigma^2$ and thus $S(f) = \sigma^2 S'(f)$ and

$$\int_0^\infty S'(f)df = 1 \tag{5-20}$$

Logarithmic spectrum is $S^\ell = fS(f)/\sigma^2$, it is dimensionless and because $d\log_e f = \frac{1}{f}df$,

$$\int_0^\infty \frac{fS(f)}{\sigma^2} d\log_e f = 1 \tag{5-21}$$

The relationship between $S'(f)$ and $S^\ell(f)$ is shown in Fig. 5.9.

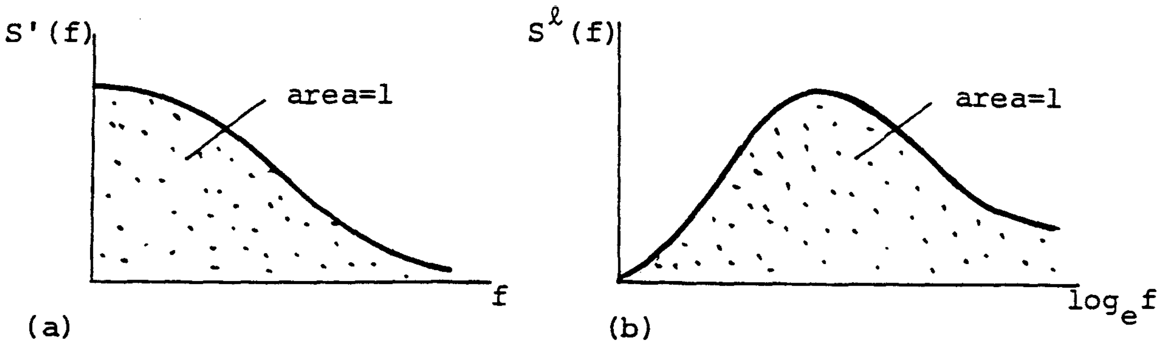


Fig. 5.9 Relationship between (a) - normalized spectrum and (b) - logarithmic spectrum

When the process $x(t) = const.\, x'(t)$, the spectrum is, with regard to equations (5-13) and (5-15)

$$S_x(f) = (Const)^2\, S'_x(f) \tag{5-21a}$$

$$S_P(f) = m^2 S_{\ddot{y}_g}(f) \tag{5-22}$$

where $S_{\ddot{y}_g}(f)$ is the spectrum of ground acceleration.

Examples of Random Processes. - Examples of typical random processes and their comparison with a deterministic harmonic process are shown in Fig. 8.10. Mathematical expressions for some correlation functions and the corresponding power spectra can be found in Ref. 10. Power spectra of a few earthquake ground motions are plotted in Fig. 5.11.

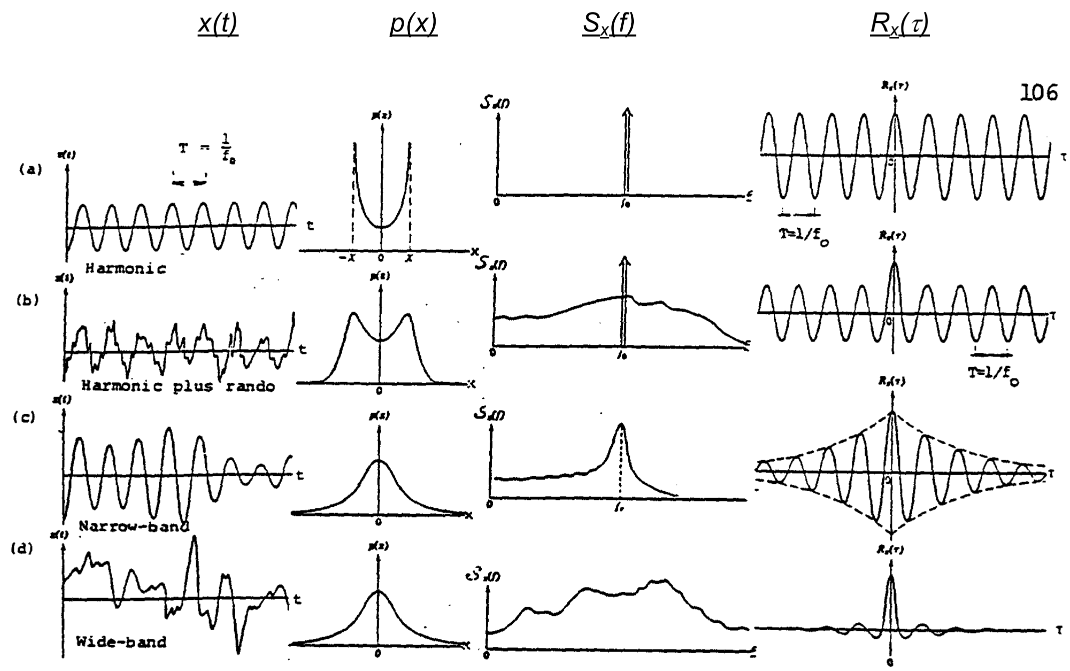
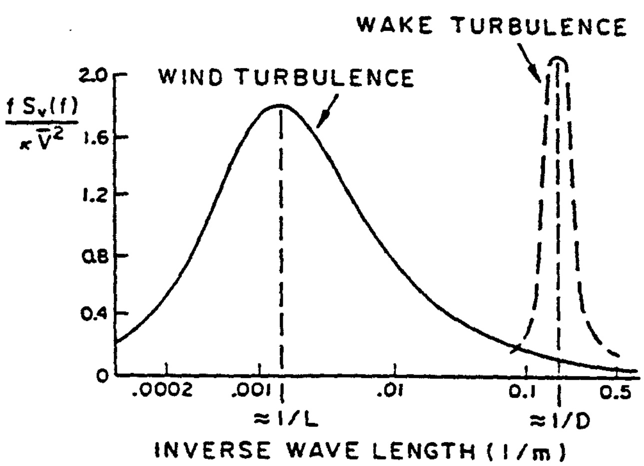


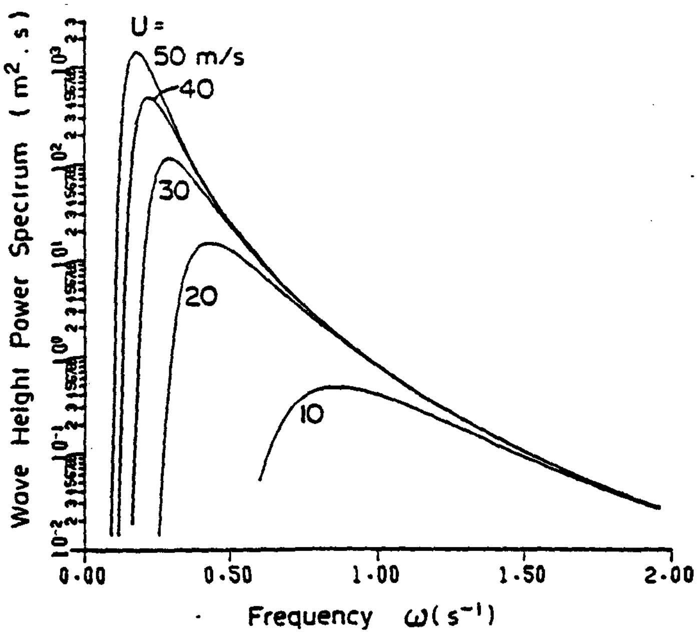
Fig. 5.10 Typical random processes and their description in terms of probability density, power spectral density and autocorrelation function.



$$S_V(f) = 4K\bar{V}_{10}^2 \frac{L/\bar{V}_{10}}{(2+\bar{f}^2)^{5/6}}$$

K = surface roughness factor = 0.08 to 0.14,
L = scale length = 1200m,
$\bar{V}_{10}$ = mean wind velocity 10 m above ground in m/s and $\bar{f} = fL/\bar{V}_{10}$.

Fig. 5.10a Power spectral density of wind velocity fluctuations $S_V(f)$ and schematic of fluctuations in wake (Davenport, Harris)



$$S(\omega) = \frac{\alpha g^2}{\omega^5} \exp[-0.74(\omega_o / \omega)^4]$$

where:
α = 8.1 x 10^{-3},
g = 9.81 m/s ^{2} and
ω_o = g/U = frequency of spectrum peak

Fig. 5.10b Pierson-Moskowitz spectrum of sea surface elevation as function of wind speed *U*.

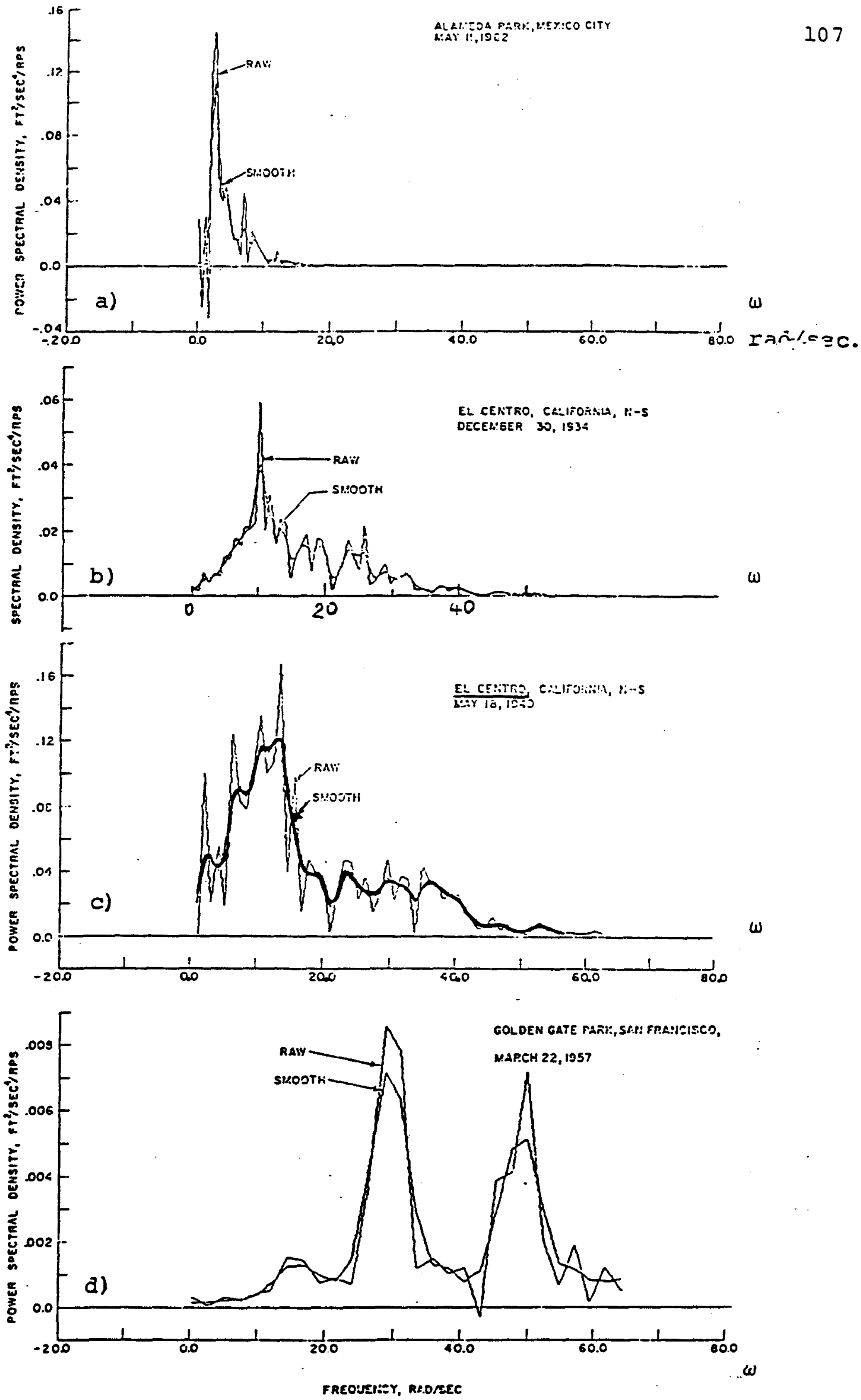


Fig. 5.11 . Power Spectra of Earthquake Ground Acceleration

5.3 Response to Random Load in one Degree of Freedom

Relationship between Input and Output

A *periodic* force P(t) with period $T = 1/f_1$ can be represented by a complex Fourier series as:

$$P(t) = \sum_{-\infty}^{\infty} c_r e^{ir2\pi f_1 t}\ , \quad c_r = \frac{1}{T}\int_{-T/2}^{T/2} P(t) e^{-ir2\pi f_1 t} dt\ ,\ r\text{=1,2 …..} \qquad \text{(5-23)}$$

The response of a SDF system to such a load can be obtained by means of superposition of responses to individual components *r* in terms of the frequency response function (admittance). The harmonic load

$$P(t) = P_o e^{i\omega t}$$

yields response

$$y(t) = \frac{P_o}{k} H(\omega) e^{i\omega t}$$

where

$$H(f) = \frac{1}{1 - \left(\frac{f}{f_o}\right)^2 + i2D\frac{f}{f_o}} \qquad \text{(5-24)}$$

However, the response to a series of harmonic loads (equation 5-23)

$$y(t) = \sum_{-\infty}^{\infty} \frac{1}{k} c_r H(f_r) e^{ir2\pi f_1 t}\ ,\ \text{with}\ f_r = r f_1 \qquad \text{(5-25)}$$

The *mean square* response can be expressed in terms of Parceval's

$$\overline{y^2} = \sum_{-\infty}^{\infty} |c_r|^2 \qquad \text{(5-26)}$$

because (5-25) is again a Fourier series with amplitudes $\frac{1}{k} c_r H(f_r)$. Hence

$$\overline{y^2} = \frac{1}{k^2} \sum_{-\infty}^{\infty} |c_r|^2 |H(f_r)|^2 \qquad \text{(5-27)}$$

realizing that

$$|y_1 y_2|^2 = |y_1|^2 |y_2|^2$$

A non-periodic (random) force can only be expressed in the above manner if period T is extended to ∞, thus from (5-27)

$$\overline{y^2} = \frac{1}{k^2} \lim_{T\to\infty} 2\sum_0^\infty |c_r|^2 |H(f_r)|^2 \qquad (5\text{-}28)$$

as $\left|{c_r}^2\right|$ and $|H(f_r)|^2$ are even functions. Substitution for c_r from 5-23 gives

$$\overline{y^2} = \frac{1}{k^2} \lim_{T\to\infty} \sum_0^\infty \frac{2}{T^2} \left| \int_{-T/2}^{T/2} P(t) e^{-ir2\pi f_1 t} dt \right|^2 |H(f_r)|^2$$

With period $T \to \infty$

$$1/T \to df \;,\; \sum_0^\infty \to \int_0^\infty ,\; \int_{-T/2}^{T/2} P(t) e^{-ir2\pi f_1 t} dt \to A(if) \;\text{ as }\; rf_1 = f_r \to f \;\text{ and }\; H(f_r) \to H(f)$$

Also, the mean square response can be expresses by means of its power spectrum, $\overline{y^2} = \int_0^\infty S_y(f) df$. Hence:

$$\int_0^\infty S_y(f) df = \frac{1}{k^2} \int_0^\infty \lim_{T\to\infty} \left[\frac{2}{T} |A(if)|^2 \right] |H(f)|^2 df$$

As

$$\lim_{T\to\infty} \left[\frac{2}{T} |A(if)|^2 \right] = S_p(f)$$

i.e. the spectrum of the excitation *P(t)*, the relation between the spectrum of the input and the spectrum of the output is:

$$\boxed{S_y(f) = \frac{1}{k^2} S_p(f) |H(f)|^2} = |\alpha(if)|^2 S_p(f) \qquad (5\text{-}29)$$

in which $|H(f)|^2$ is the square of the modulus of the admittance function, which is equal to the square of the dynamic magnification factor and is:

$$|H(f)|^2 = \varepsilon^2 = \frac{1}{\left[1-(f/f_o)^2\right]^2 + 4D^2(f/f_o)^2} \tag{5-30}$$

The relationship between the input and the output described by equation 5-29 is shown in Fig. 5.12.

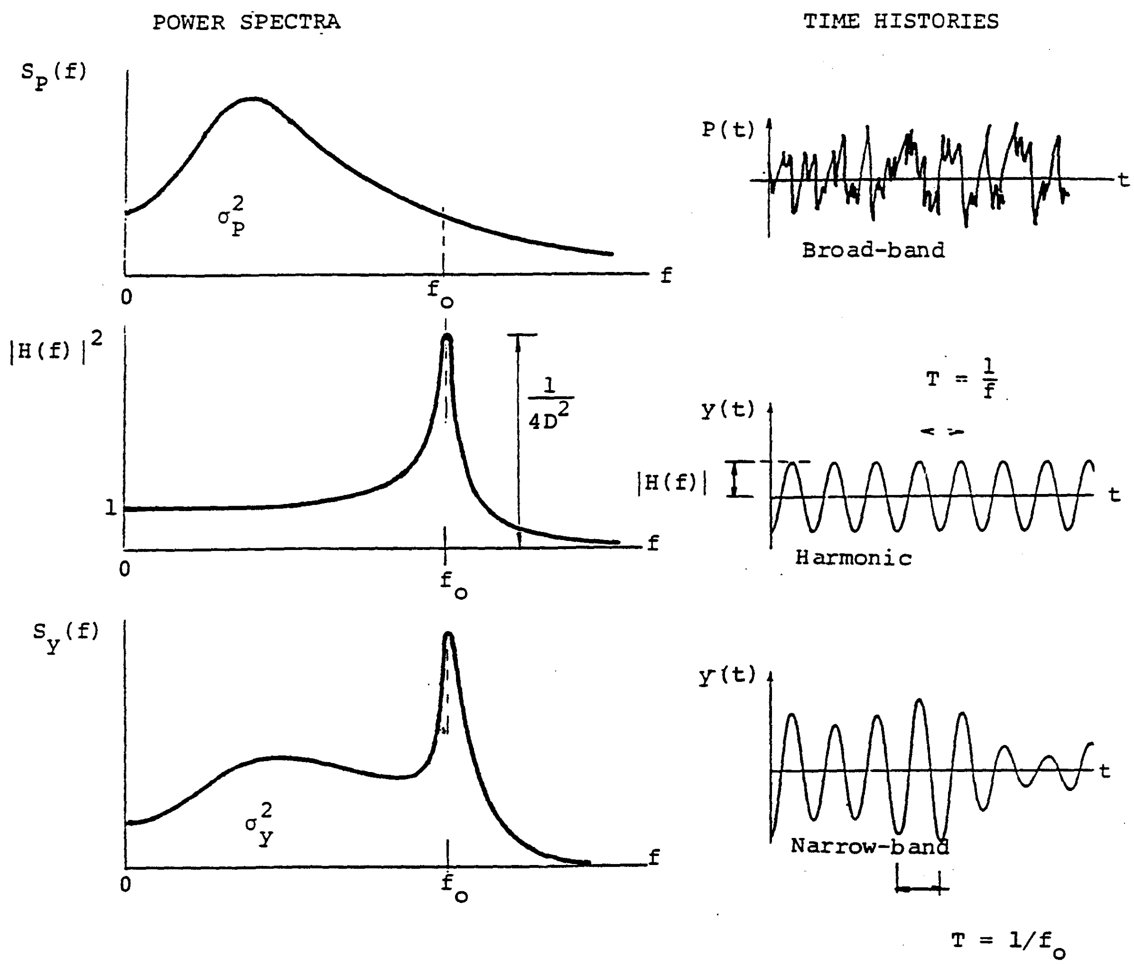


Fig. 5.12 The relationship between spectrum of input and spectrum of output

With response spectrum defined by equation 5-21, the mean square response $\overline{y^2}$ is

$$\overline{y^2} = \int_0^\infty S_y(f)df = \frac{1}{k^2}\int_0^\infty S_p(f)\left|H(f)\right|^2 df \qquad (5\text{-}31)$$

or in terms of circular frequency

$$\overline{y^2} = \int_0^\infty 2\pi S_y(\omega)\frac{d\omega}{2\pi} = \int_0^\infty S_y(\omega)d\omega$$

From equation 5-31 the rms displacement having the dimension of amplitudes, is $\sqrt{\overline{y^2}} = \sigma_y$. The only complication is that the integral in equation 5-31 cannot be generally evaluated in closed form. It can be evaluated approximately as:

$$\begin{aligned}\overline{y^2} &\cong \frac{1}{k^2}\int_o^{f_o} S_p(f)df + \frac{1}{k^2}\int_0^\infty S_p(f_o)\left|H(f)\right|^2 df \\ &= \frac{1}{k^2}\int_0^{f_o} S_p(f)df + \frac{1}{k^2}S_p(f_o)\frac{\pi f_o}{4D}\end{aligned} \qquad (5\text{-}32)$$

This approximate evaluation is based on replacing $\left|H(f)\right|^2$ by unity for frequencies from 0 to f_o and on replacing the force power spectrum $S_P(f)$ by a constant (white) spectrum $S_P(f_o)$ whose magnitude is equal to the force spectrum for the natural frequency of the system, f_o (Fig. 5.13). The first part of equation 5-32 is called the background effect and the second part the resonant effect.

If greater accuracy is needed, the integral in equation 5-31 can be evaluated using the theory of residua or numerical integration.

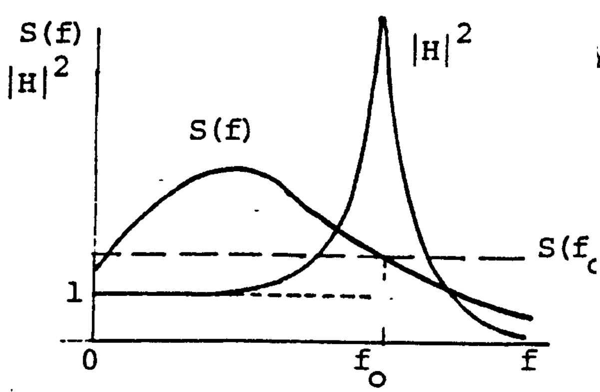


Fig. 5.13 Approx. evaluation of response

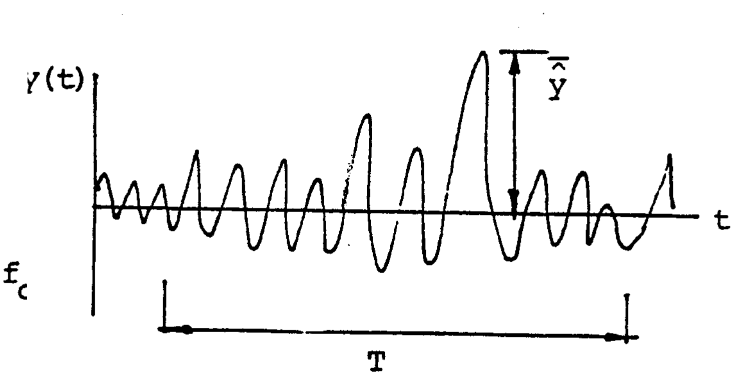


Fig. 5.14 Peak of random response

When the damping is small and the spectrum flat, the second part of equation 5-32 yields sufficient accuracy and thus, the variance of the response is approximately

$$\overline{y^2} \cong \frac{1}{k^2} S_P(f_o) \frac{\pi}{4} \frac{f_o}{D} = \frac{1}{k^2} 2\pi S_P(\omega_o) \frac{\pi}{4} \frac{\omega_o}{2\pi D} = \frac{1}{k^2} S_P(\omega_o) \frac{\pi}{4} \frac{\omega_o}{D} \qquad (5\text{-}33)$$

The input spectrum and the system natural frequency can be expressed in terms of frequency f (Hz) or ω (rad/sec). In both cases the formulae are formally the same.

From the variance $\overline{y^2}$ the standard deviation (root-mean-square) of the response follows as

$$\sigma_y = \sqrt{\overline{y^2}}$$

The r.m.s. response depends on the square root of the damping ratio.

The standard deviation determines the distribution of all values of the response, as can be seen from equation 5-11 and Fig. 5.6, but the peak, i.e. maximum value of the response indicated in Fig. 5.14 is of primary importance for design.

Peak Value of Response

During each period of observation T, one largest (peak) value of the response can be established. This largest value depends on the duration of the observation, T, and the apparent frequency ν, which depends on the spectrum of the process and is:

$$\nu = \sqrt{\frac{\int_0^\infty f^2 S(f) df}{\int_0^\infty S(f) df}} \quad \text{(Hz)} \qquad (5\text{-}34)$$

For a narrow band process such as the response of a lightly damped system, the apparent frequency v is close to the natural frequency and thus, $\nu \cong f_o$. The peak values observed in individual observations may be assembled to yield a probability density distribution (Fig. 5.15). The mean value of the peaks can be evaluated as:

$$\bar{\hat{y}} \cong g\sigma_y \qquad (5\text{-}35)$$

in which the peak factor $g = \bar{\hat{y}} / \sigma_y$ can be calculated using the formula:

$$g = \sqrt{2\log_e \nu T} + \frac{0.5772}{\sqrt{2\log_e \nu T}} \qquad (5\text{-}36)$$

The peak factor ranges between about 2.5 and 4.5 (Fig. 5.16).
(see: Davenport, A.G., “The Distribution of the Largest Values of a Random Function With Application to Gust Loading”, Proc. ICE, Vol. 28., No. 6739, June 1964, pp 187-196 ….. and
Rice, S.O. "Mathematical Analysis of Random Noise," Selected Papers on Noise and Stochastic Processes, edited by N. Wax, Dover Publ., New York, 1954.)

Response to earthquakes. - With regard to equations 5-22 and 5-33, the variance of earthquake response is

$$\overline{y^2} = \frac{1}{k^2}\frac{\pi}{4}\frac{\omega_o}{D} m^2 S_{\ddot{y}_g}(\omega_o) \qquad (5\text{-}37)$$

A more accurate analysis should consider nonstationarity but the assumption of stationarity is conservative.

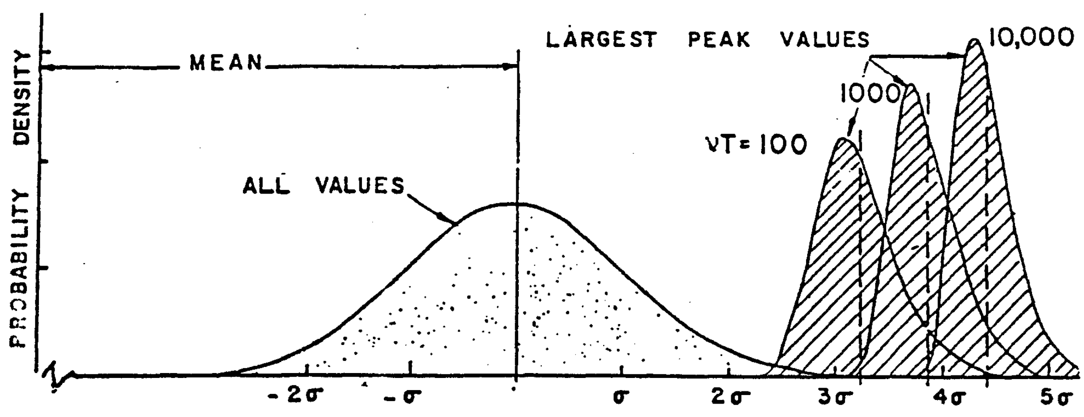


Fig. 5.15 Probability distributions of all values and peak values

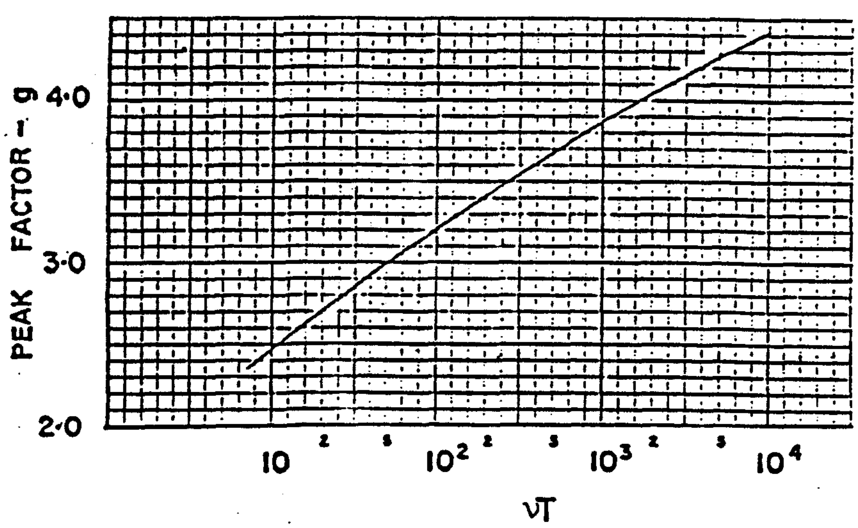
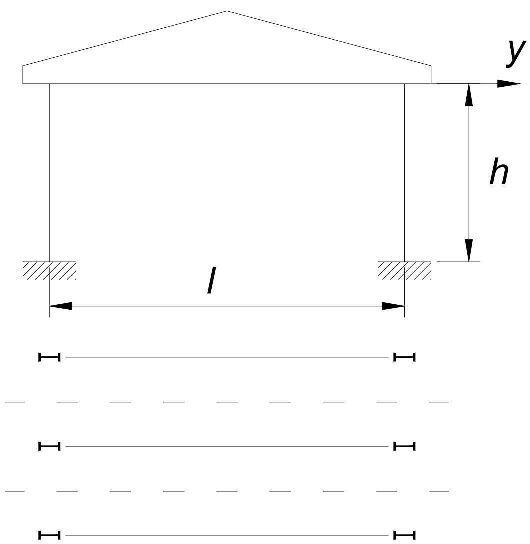


Fig. 5.16

Peak Factor vs. νT

Problem 5.1: Predict the seismic response of the one storey shear building given in below to the El Centro 1940 earthquake in terms of random vibration. The power spectrum of that earthquake is given in Fig. 5.llc. Assume damping ratio D = 2% and strong motion duration T = 30 s. (In Fig. 5.llc, the power spectrum corresponds to the original peak ground acceleration of 0.3 g.)



Each section of shear building is supported by two columns having a depth, d of 600 mm

$I = 560 \times 10^6$ mm^4
$E = 2.0 \times 10^5$ Mpa
h = 5.0 m

The participating mass of the structure is:

m = 30,000 kg (for one bay)

5.4 RESPONSE OF MULTI-DEGREE -OF-FREEDOM SYSTEMS TO RANDOM LOADING

5.4.1 Fully Correlated Load

The motion of the ground or the forces acting directly upon masses m_i are often random. If the forces have the same time history (phase shift) at each mass but different amplitudes they are *fully correlated.* This is the case with ground excitation when the effective forces are:

$$P_i(t) = (-)m_i \ddot{u}_g(t)$$

or with direct excitation,

$$P_i(t) = P_i f(t)$$

where $\ddot{u}_g(t)$ or $f(t)$ are common for all masses. An example is a large wind gust hitting a relatively small structure (Fig. 5.17). The response is again given by equation 4.2, i.e.

$$u_i(t) = \sum_j \Phi_{ij}\eta_j \qquad (5\text{-}38)$$

in which η_j is given by equation (5-11),

$$M_j\ddot{\eta}_j + 2M_jD_j\omega_j\dot{\eta}_j + K_j\eta_j = L_jf(t)$$
$$= L_j\ddot{u}_g(t) \qquad (5\text{-}39)$$

in which, for ground excitation

$$L_j = \sum_i m_i\Phi_{ij} \qquad (5\text{-}40a)$$

or with external forces

$$L_j = \sum_i P_i\Phi_{ij} \qquad (5\text{-}40b)$$

and

$$K_j = \omega_j^2 M_j$$

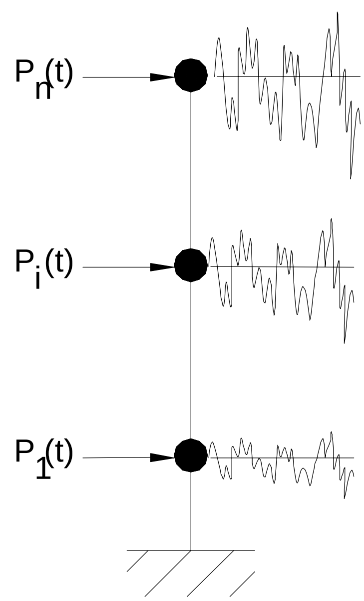
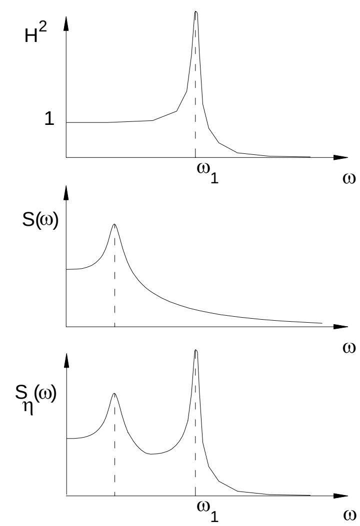


Fig. 5.17

If *f(t)* or $\ddot{u}_g$ is random it can be described by its power spectrum $S_{\ddot{u}_g}(\omega)$ or $S_f(\omega)$ called generally $S(\omega)$. By equation 5-21a), the power spectrum of the right side of equation.(5-39) is

$$S_j(\omega) = L_j^2 S(\omega) \qquad (5\text{-}41)$$

Equation (5-39) is an equation of SDF system and therefore the spectrum of coordinate η_j is by equation (5-29)

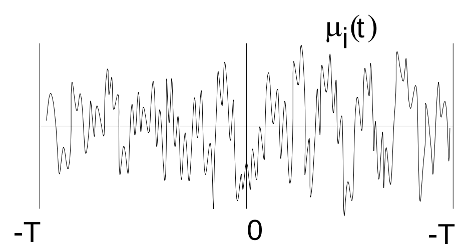


$$S_{\eta_j}(\omega) = \frac{1}{K_j^2} L_j^2 S(\omega) \left| H\left(\frac{\omega}{\omega_j}\right)\right|^2 \qquad (5\text{-}42)$$

where the square of the mode of the mechanical admittance

$$\left| H\left(\frac{\omega}{\omega_j}\right)\right|^2 = \frac{1}{\left(1-\frac{\omega^2}{\omega_j^2}\right) + 4D^2\left(\frac{\omega}{\omega_j}\right)^2}$$

Variance of η_j becomes:



$$\overline{\eta_j^2} = \frac{L_j^2}{K_j^2}\int_0^\infty S(\omega)\left| H\left(\frac{\omega}{\omega_j}\right)\right|^2 \qquad (5\text{-}43)$$

The motion is

$$u_i(t) = \sum_{j=1}^{n} \Phi_{ij}\eta_j(t)$$

or with distributed systems

$$u(x,t) = \sum_{j}^{\infty} \Phi_j(x)\eta_j(t)$$

The variance of this motion is obtained by squaring and averaging,

$$\overline{u_i^2} = \frac{1}{\underset{T\to\infty}{2T}} \int_{-T}^{T} \left(\sum_j \Phi_{ij} \eta_j(t) \right)^2 dt = \frac{1}{\underset{T\to\infty}{2T}} \int_{-T}^{T} \left(\sum_j \Phi_{ij}^2 \eta_j^2(t) + \sum \text{product terms} \right) dt$$
$$= \frac{1}{\underset{T\to\infty}{2T}} \int_{-T}^{T} \left(\sum_{r=1}^{n} \sum_{s=1}^{n} \Phi_{ir} \Phi_{is} \eta_r \eta_s \right) dt = \sum_{r=1}^{n} \sum_{s=1}^{n} \overline{\eta_r \eta_s} \Phi_{ir} \Phi_{is} \tag{5-44}$$

The cross products between the generalized coordinates complicate the situation. However, they can be neglected if

1) Natural frequencies are well separated
2) Damping is small or at least not very large.

Then

$$\overline{u_i^2} = \frac{1}{\underset{T\to\infty}{2T}} \int_{-T}^{T} \left(\sum_j \Phi_{ij}^2 \eta_j^2(t) \right) dt = \sum_j \Phi_{ij}^2 \frac{1}{2T} \int_{-T}^{T} \eta_j^2(t) dt$$

As $\frac{1}{2T} \int_{-T}^{T} \eta_j^2(t) dt = \overline{\eta_j^2}$ = the variance of generalized coordinate as given by Equation (5-43), the resonance of the displacement of mass m_i is

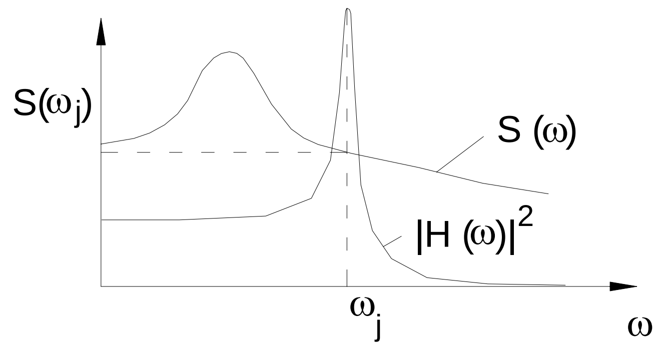
$$\overline{u_i^2(t)} = \sum_{j=1}^{n} \Phi_{ij}^2 \overline{\eta_j^2} = \sum_{j=1}^{n} \Phi_{ij}^2 \frac{L_j^2}{K_j^2} \int_0^{\infty} S(\omega) \left| H\left(\frac{\omega}{\omega_j} \right) \right|^2 d\omega \tag{5-45}$$

The integral in equation (5-45) can be evaluated by means of the theory of residua or numerically as already discussed. If damping is small and the spectrum rather flat, an approximate solution indicated in equation (5-33) and Fig. 5.13 is usually sufficiently accurate, i.e.

$$\int_0^{\infty} S(\omega) |H(\omega)|^2 d\omega \cong S(\omega) \int_0^{\infty} |H(\omega)|^2 d\omega = S(\omega_j) \frac{\pi}{4} \frac{\omega_j}{D} \tag{5-46}$$

With this approximation, equation (5-45) simplifies and the variance of the displacement is

$$\overline{u_i^2(t)} = \frac{\pi}{4D}\sum_{j=1}^{n}\Phi_{ij}^2\frac{L_j^2}{K_j^2}S(\omega_j)\omega_j$$



Substituting for $K_j = \omega_j^2 M_j$, it is also

$$\overline{u_i^2(t)} = \frac{\pi}{4D}\sum_{j=1}^{n}\Phi_{ij}^2\frac{L_j^2}{M_j^2}\frac{S(\omega_j)}{\omega_j^3} = \frac{1}{64\pi^3 D}\sum_{j=1}^{n}\Phi_{ij}^2\frac{L_j^2}{M_j^2}\frac{S(f_j)}{f_j^3} \qquad (5\text{-}47)$$

The RMS displacement is $\sigma_{u_i} = \sqrt{\overline{u_i^2(t)}}$. Only one, two or three first modes usually need to be considered. Very often, one mode is enough (the first or the second), e.g. for buildings exposed to wind gusts or earthquake excitation. Max. (peak) values follow from equation (5-35) and range from 3.5 to 4.5 RMS.

If the power spectrum is available as a function of frequency *f*, it is possible to use either one as

$$S(\omega_j) = \frac{1}{2\pi}S(f_j)$$

If the damping ratio, *D*, is assumed to be different for each vibration mode, it remains as D_j behind the summation sign $\sum$ in equation (5-47).

Problem 5.2: Analyze the response of the five storey shear building to earthquake excitation defined by the power spectrum shown in Fig. 5.11c (El Centro, 1940). Calculate:

(a) Peak response in individual modes u_{ij} assuming the duration of the strong motion T = 30 s.
(b) Equivalent seismic forces $q_{ij} = m_i\omega_j^2\hat{u}_{ij}$
(c) Compare the results with those obtained by means of the pseudovelocity spectrum.

5.4.2 Partially Correlated Loads

When the loads P_i acting at individual stations of a structure (Fig. 5.17) are not fully correlated, their total effect on the response in the fundamental mode is reduced. This reduction is very significant in the case of wind loading as is discussed in Chapter 6. The analysis requires a greater amount of input information and is more difficult but can lead to useful observations (see: Novak, M. "Random Vibration of Structures", Proc. 4th Intern. Conference on Application of Statistics and Probability in Soil and Structural Engineering, Florence, 1983, pp. 539-550).

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