Review of Modal Analysis Concepts

- A method which describes the response of a dynamic system by the summation of its response in its various orthogonal modes of vibration.

STEP1

- Carry out a "eigenvalue analysis" of the simultaneous equations which describes the free vibration of the structure. This gives
  a) Eigenvalues or natural frequencies
  b) Eigenvectors or mode shapes of the modes of vibration

STEP2

- Solve equations of motion and determine response due to a given type of excitation

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\} = \{P\} \qquad (1)$$

$[m]$, $[c]$ and $[k]$ are mass, damping and stiffness matrices
$\{u\}$ = the displacement vector
$\{P\}$= the vector of excitation.

- The response of the system is evaluated one mode at a time, then summed over all modes Φ_{ij},

$$u_i(t) = \sum_{j=1}^{n} \Phi_{ij}\eta_j(t), \quad i = 1,2 \ldots n \qquad (2)$$

in which:

- $u_i(t)$ is the total response at location i as a function of time
- Φ_{ij} are modal coordinates of the j^{th} mode.
- $\eta_j(t)$ are the generalized coordinates describing the magnitude and time dependence of the response in each mode.

- Eq. 2 can be rewritten in matrix form to include all nodes, *i = 1, 2..., n:*

$$\{u\} = [\Phi]\{\eta\}, \qquad \{\dot{u}\} = [\Phi]\{\dot{\eta}\}, \qquad \{\ddot{u}\} = [\Phi]\{\ddot{\eta}\} \tag{3}$$

where

$$[\Phi] = \left[\begin{array}{c|c|c|c} \Phi_{11} & \Phi_{12} & \ldots & \Phi_{n1} \\ \Phi_{21} & \Phi_{22} & \ldots & \Phi_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \Phi_{n1} & \Phi_{n2} & \ldots & \Phi_{nn} \end{array}\right] \tag{4}$$

j =   *1*   *2*   *…*   *n*

- Each column in Eq. 4 represents one mode of free vibrations that is obtained from the Eigenvectors (modes).
- $\{\eta\}$ is the vector of generalized coordinates of j^{th} mode

- The equations of motion under the action of load $\{P(t)\}$ are

$$\underbrace{[\Phi]^T[m][\Phi]}_{[M^*]}\{\ddot{\eta}\} + [\Phi]^T[c][\Phi]\{\dot{\eta}\} + \underbrace{[\Phi]^T[k][\Phi]}_{[K^*]}\{\eta\} = [\Phi]^T\{P(t)\} \tag{5}$$

The *Generalized Mass (by orthogonality)*   *The Generalized Stiffness* (*by orthogonality)*

- The damping term also becomes a diagonal matrix only when the damping matrix, [*c*], is proportional to either

$[c] = 2\alpha[m]$ then $[\Phi]^T[c][\Phi] = [\Phi]^T 2\alpha[m][\Phi] = 2\alpha[M^*]$

or

$[c] = \beta[k]$ then

$$[\Phi]^T[c][\Phi] = [\Phi]^T \beta[k][\Phi] = \beta[\Phi]^T\left[\omega_j^2\right][m][\Phi] = \beta\left[\omega_j^2\right][M^*]$$

If the damping terms can be uncoupled, then we have *n* independent equations of the form

$$M_j\ddot{\eta}_j + 2\alpha M_j\dot{\eta}_j + \omega_j^2 M_j\eta_j = \{\Phi_j\}^T\{P(t)\} \qquad (11)$$

or

$$\ddot{\eta}_j(t) + 2\alpha\dot{\eta}_j(t) + \omega_j^2\eta_j(t) = \frac{p_j(t)}{M_j}, \quad j = 1,2,3 \ \dots \ n \qquad (12)$$

- where

$$p_j(t) = \{\Phi_j\}^T\{P(t)\} = \sum_{i=1}^{n}\Phi_{ij}P_i(t) \qquad (13)$$

  is the generalized force for mode *j*

- Solve the “n” equations each in only one generalized coordinate, one at a time

7.1 HARMONIC EXCITATION

- Assume harmonic excitation with frequency was in the case of un-balanced masses of machines, vortex shedding etc. Such forces can be described as:

$$P_i(t) = P_i\cos\omega t\text{, or } \{P(t)\} = \{P\}\cos\omega t \qquad (14)$$

- The generalized forces for mode *j,* are from Eq. 13

$$p_j(t) = \cos\omega t\underbrace{\sum_{i=1}^{n}\Phi_{ij}P_i}_{\substack{\text{the force}\\\text{participation}\\\text{factor, } L_j}} \qquad p_j(t) = L_j\cos\omega t \qquad (15\ \&\ 16)$$

the *Force Participation Factor* in mode *j*

- The generalized equation of motion, Eq. 12, is

$$\ddot{\eta}_j(t) + 2\alpha\dot{\eta}_j(t) + \omega_j^2\eta_j(t) = \frac{L_j}{M_j}\cos\omega t \qquad (17)$$

- The solution of Eq. 17 follows from the SDOF solution found previously,

<u>Steady State Response</u>

$$\eta_j(t) = \underbrace{\eta_j \cos(\omega t + \phi_j)}_{\text{Steady State}} \qquad (18)$$

- As in the SDF system, the amplitude becomes:

$$\eta_j(t) = \frac{L_j}{M_j\omega_j^2}\varepsilon_j = \frac{L_j}{K_j}\varepsilon_j = (\eta_{st})_j\,\varepsilon_j$$

- The total response or steady motion at location, *i* is:

$$u_i(t) = \sum_{j=1}^{n}\Phi_{ij}\eta_j = \sum_{j=1}^{n}u_{ij}\cos(\omega t + \phi_j) \qquad (21)$$

where the amplitude in mode *j* is

$$u_{ij} = \Phi_{ij}\eta_j = \Phi_{ij}\frac{L_j}{M_j\omega_j^2}\varepsilon_j$$

- the phase shifts in each mode are:

$$\phi_j = -\tan^{-1}\frac{2D_j\,\omega/\omega_j}{1-\left(\omega/\omega_j\right)^2}$$

7.2 RESPONSE TO GROUND MOTION

- Analogous to SDF equations of motion can be written in terms of each generalized coordinate

$$\ddot{\eta}_j + 2\alpha\dot{\eta}_j + \omega_j^2\eta_j = \frac{p_j(t)}{M_j} = \frac{L_j}{M_j}\ddot{u}_g(t) \qquad (26)$$

$$L_j = \sum_{i=1}^{n} m_i \Phi_{ij}$$ = Earthquake Participation Factor

$$M_j = \sum_{i=1}^{n} m_i \Phi^2{}_{ij}$$ = Generalized Mass

- Determine the response in each mode using the Duhamel Integral, as in the case of SDF

$$\eta_j(t) = \frac{1}{\omega_j}\frac{L_j}{M_j}\int_0^t \ddot{u}_g(\tau)\, e^{-D_j\omega_j(t-\tau)}\sin\omega_j(t-\tau)d\tau$$

$$= \frac{1}{\omega_j}\frac{L_j}{M_j}V_j(t)$$

as in SDF, $V_{j,\max} = S_v(j)$, the Spectral Velocity for mode "*j*"

- As in SDF:

$$S_d = \frac{S_v}{\omega_j}, S_a = \omega_j S_v$$

$$\hat{u}_{ij} = \frac{L_j}{M_j}\Phi_{ij}S_{d(j)}$$

Force at m_i : $$q_{ij} = m_i\frac{L_j}{M_j}\Phi_{ij}S_{a(j)}$$

Total Shear: $$Q_j = \sum_{i=1}^{n} q_{ij} = \frac{L_j}{M_j}S_{a(j)}\sum_{i=1}^{n} m_i\Phi_{ij} = \frac{L_j^2}{M_j}S_{a(j)}$$

- Total Response obtained by summing the responses of all “ *n* ” modes

$$\hat{u}_{i,\max} \cong \sqrt{\sum_j \hat{u}_{i,j}^2}$$

- All responses (force, moment, acceleration or displacement) are evaluated in the same way. The responses in the various modes are independent and <u>the square of the total response is equal to the sum of the squares of the responses in each mode.</u>