Review of Random Loading Concepts - II

Assuming a periodic signal with the form: $x(t) = a\sin\omega t$

The mean is:
$$\overline{x(t)} = \overline{x} = \underset{T\to Large}{\frac{1}{T}} \int_0^T a\sin\omega\, t\, dt = 0$$

The variance is:
$$\overline{x^2(t)} = \overline{x^2} = \sigma_x^2 = \underset{T\to Large}{\frac{1}{T}} \int_0^T \left(x(t) - \overline{x}\right)^2 dt$$
$$= \frac{1}{T}\int_0^T a^2 \sin^2 \omega\, t\, dt$$
$$= \frac{a^2}{T}\int_0^T \frac{1}{2}(1 - \cos 2\omega\, t)\, dt$$
$$= \frac{a^2}{T}\frac{T}{2} = \frac{a^2}{2}$$

The Standard Deviation (RMS) is: $\sigma_x = \sqrt{\overline{x^2}} = \frac{1}{\sqrt{2}}a$

The Peak Value is: $x(t)\big|_{max} = \hat{x} = a = \sqrt{2}\sigma_x$

The $\sqrt{2}$ is a "Peak factor" $g = \sqrt{2}$

Identical expressions result from assuming a periodic signal of the form:
$x(t) = a\cos\omega t$

The Probability Density Function, $p_x(x)$ is:

the probability that $x_1 \le x \le x_1 + dx = p_x(x_1)dx$

$= \frac{d\theta_{x=x_1}}{\theta}$ or the proportion of time between $\theta \text{ and } \theta_1$ over the total time

where $\theta = \sin^{-1}(x/a)$

$$d\theta = \frac{d(x/a)}{\sqrt{1 - x^2/a^2}} = \frac{dx}{a\sqrt{1 - x^2/a^2}} = \frac{dx}{\sqrt{a^2 - x^2}}$$

for $x_1 \ge 0$, $0 \le \theta \le \pi$

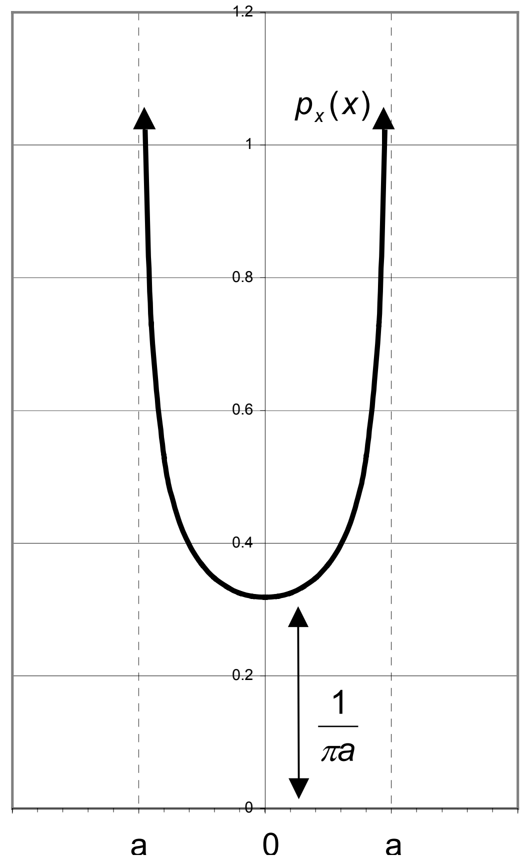
$$p_x(x_1)dx = \frac{d\theta}{\pi} = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x_1^2}}$$

for $x_1 \le 0$, $\pi \le \theta \le 2\pi$

$$p_x(x_1)dx = \frac{d\theta}{\pi} = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x_1^2}}$$

Therefore; $p_x(x) = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x^2}}$

At $x = 0$; $p_x(x) = \frac{1}{\pi a}$, and at $x = a$; $p_x(x) \to \infty$,



The Autocorrelation Function is:

$$R(\tau) = \lim_{T \to large} \frac{1}{T} a^2 \int_0^T \sin(\omega t + \phi)\sin(\omega(t+\tau) + \phi)dt$$

$$= \frac{1}{T} a^2 \int_0^T \sin(\omega t + \phi)[\sin(\omega t + \phi)\cos\omega\tau + \cos(\omega t + \phi)\sin\omega\tau]dt$$

$$= \frac{a^2}{T}\int_0^T \sin^2(\omega t + \phi)\cos\omega\tau dt + \frac{a^2}{T}\int_0^T \sin(\omega t + \phi)\cos(\omega t + \phi)\sin\omega\tau dt$$

(due to orthogonality of sin & cos = 0)

Therefore: $R(\tau) = \frac{a^2}{T}\int_0^T \frac{1}{2}(1 - \cos 2(\omega t + \phi))\cos\omega\tau dt$

$$= \frac{a^2}{T}\frac{T}{2}\cos\omega\tau = \frac{a^2}{2}\cos\omega\tau$$

The normalized autocorrelation function is normalized by the variance, σ^2

$$R'(\tau) = \frac{R(\tau)}{\sigma^2} = \frac{(a^2/2)\cos\omega\tau}{a^2/2} = \cos\omega\tau$$

Relationship Between Input and Output

The mean-square response of the output (i.e. response) was shown in Eq. 8.32 to be composed of two parts; the first attributable to the quasi-steady “Background” component and the other to the “Resonant”:

$$\begin{aligned}\overline{y^2} &\cong \frac{1}{k^2}\int_o^{f_o} S_p(f)df + \frac{1}{k^2}\int_0^{\infty} S_p(f_o)\left|H(f)\right|^2 df \\ &= \frac{1}{k^2}\int_0^{f_o} S_p(f)df + \frac{1}{k^2} S_p(f_o)\frac{\pi f_o}{4D}\end{aligned} \tag{8.32}$$

A similar, but more convenient approximation to this expression recognizes the fact that the integral of the input spectrum from 0 up to the natural frequency f_o is approximately equal to the integral over the entire spectrum of input, when f_o is reasonably high, compared with the dominant frequencies of the input spectrum. This is the case for most applications involving wind, for example. The equation for the variance reduces to:

$$\begin{aligned}\overline{y^2} &\cong \frac{1}{k^2}\sigma_p^2 + \frac{1}{k^2} S_p(f_o)\frac{\pi f_o}{4D} \\ &\cong \frac{\sigma_p^2}{k^2}\left[1 + \frac{\pi}{4D}\frac{f_o S_p(f_o)}{\sigma_p^2}\right]\end{aligned}$$

The Non-dimensional Force Spectrum

The Resonant Response

The Background Excitation
(a “Static” relationship between Input and Output)

$$\underline{\sigma_y^2 = \sigma_{yB}^2 + \sigma_{yR}^2}$$

or, the variance of the Total Response equals the sum of the variance of the Background Response plus the Resonant response

In many situations, especially when the damping is low, $\underline{\sigma_{yB}^2 << \sigma_{yR}^2}$ and then:

$$\underline{\sigma_y^2 \approx \sigma_{yR}^2 \approx \frac{1}{k^2}\frac{\pi}{4D} f_o S_p(f_o)}$$

Properties of Power Spectra

$S_x(f_1)df$ = the contribution to the total variance of *x* (i.e. σ_x^2), due to variations of $x(t)$ with frequencies of $f_1 \le f \le f_1 + df$

$$\therefore \int_0^\infty S_x(f)df = \sigma_x^2$$

The Relationship between the PDF and the SDF

PDF – Probability Density Function, $p_x(x)$
SDF – Spectral Density Function, $S_x(f)$

If *x(t)* is Gaussian, then

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}} \qquad \text{(Eq. 8.11)}$$

where

$$\sigma_x^2 = \int_0^\infty S_x(f)df \qquad \text{(Eq. 8.18)}$$

When: $y(t) = a\, x(t)$

$S_y(f) = a^2 S_x(f)$ and $\sigma_y^2 = a^2\sigma_x^2$

As an example, with seismic excitation where the force is $p(t) = -m\ddot{y}_g(t)$,
then $S_p(f) = m^2 S_{\ddot{y}_g}(f)$.

If *x(t)* is a time-varying quantity:

$p_x(x)$ provides information on the amplitude content of *x(t)*
$S_x(f)$ provides information on the frequency content of *x(t)*
$R_x(\tau)$ provides information on the correlation of *x* with itself at time lag, τ

Description of Periodic Functions

Fourier Series – Provides information on the variation in the frequency domain at discrete harmonics of the frequency, ω, 2ω, 3ω, ……

As $T \to \infty$ this results in a Fourier Integral, providing information at all values of ω:

$$\left.\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)e^{i\omega t}\,d\omega \\ A(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\,dt \end{aligned}\right\} \text{ a Fourier Transform Pair}$$

If $\quad P(t) = P_o e^{i\omega t}$

$y(t) = \frac{P_o}{K} H(\omega) e^{i\omega t}$, the effect of the load at one harmonic of ω

then if there are many harmonic loads:

$$y(t) = \sum_{-\infty}^{\infty} \frac{1}{K} c_r H(f_r) e^{i2\pi f_1 t}$$

$$\overline{y^2} = \frac{1}{K^2} \sum_{-\infty}^{\infty} |c_r|^2 |H(f_r)|^2$$

If $T \to \infty$, $1/T \to df$, $\sum_0^{\infty} \to \int_0^{\infty}$, $rf_1 = f_r \to f$ and $H(f_r) \to H(f)$

$$\sigma_y^2 = \overline{y^2} = \int_0^{\infty} S_y(f)\,df = \frac{1}{k^2}\int_0^{\infty} S_p(f)\left|H(f)\right|^2 df$$

See Fig. 8.12 for the relationship between $S_y(f)$ and $S_p(f)$.

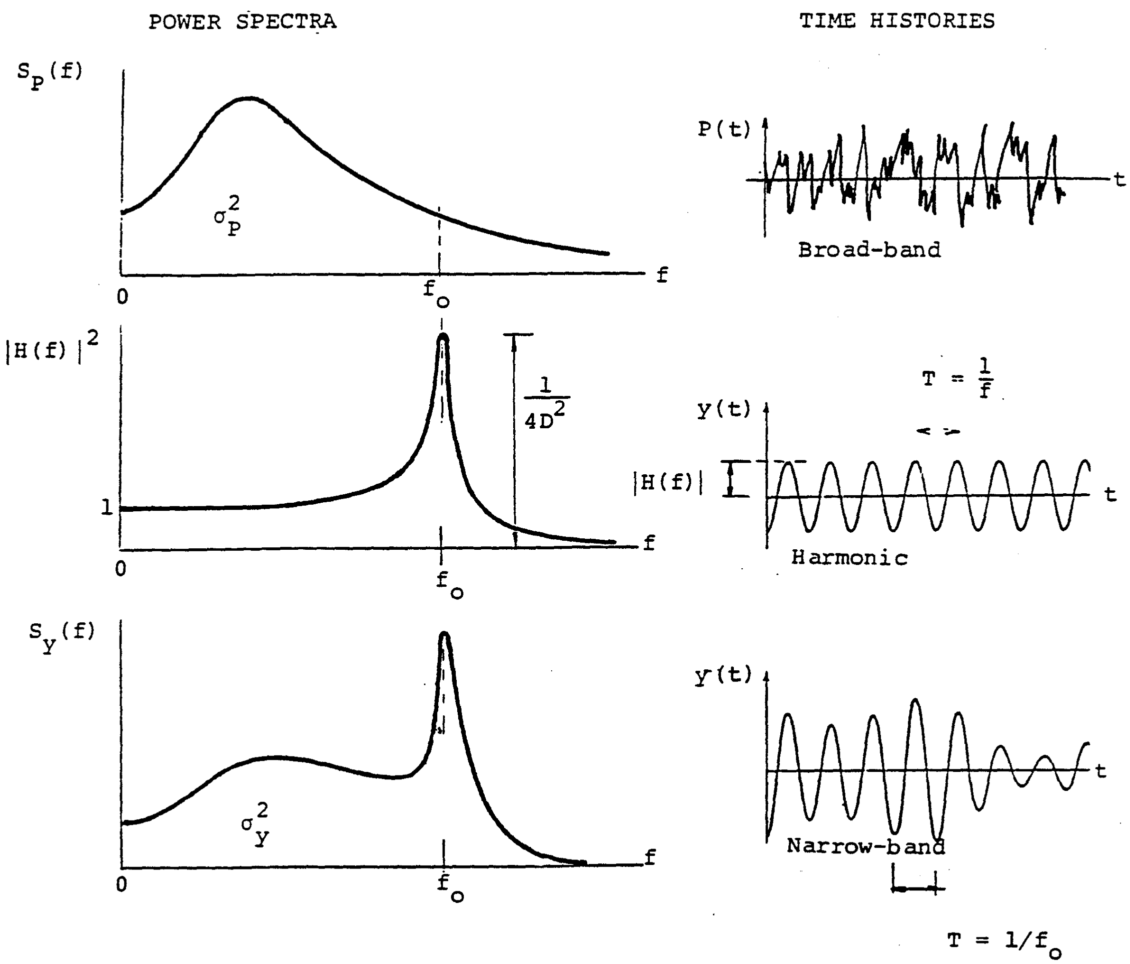


Fig. 8.12 The relationship between spectrum of input and spectrum of output

Closed form solutions of this integral are difficult, so approximations can be made with minimal loss of accuracy.

$$\sigma_y^2 = \overline{y^2} \cong \frac{1}{k^2}\int_o^{f_o} S_p(f)df + \frac{1}{k^2}\int_0^{\infty} S_p(f_o)|H(f)|^2 df$$

$$\cong \frac{1}{k^2}\int_0^{f_o} S_p(f)df + \frac{1}{k^2} S_p(f_o)\frac{\pi f_o}{4D}$$

$$\overline{y^2} \cong \frac{\sigma_p^2}{k^2}\left[1 + \frac{\pi}{4D}\frac{f_o S_p(f_o)}{\sigma_p^2}\right]$$