Review of Response to Random Loads

- We require new mathematical tools to describe "random" loads and responses of structures
- Random loads vary in:
    - Magnitude
    - Time
    - Spatially
- We use:
    - Probability Distributions
    - Power Spectra and Auto-correlations
    - Cross-Spectra and Cross-correlations

Stationary Random Process:

Strict Stationarity

- All statistical properties are invariant with time

Weak Stationarity

- Mean μ_x and autocorrelation $\rho_x(\tau)$ are invariant with time

Ergodic Process

- Stationary process where the statistical properties of one sample are identical to the ensemble statistics

- There are fully developed mathematical methods available for Ergodic Processes and thus there are enormous advantages if a process can be regarded as stationary. This includes processes that can be described as having *weak stationarity* or *local stationarity*

Auto-correlation

- $R_x(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T (x(t)-\bar{x})(x(t-\tau)-\bar{x})dt$
- $R_x(\tau) = R_x(-\tau)$ (an "even" function)
- $R_x(\tau=0) = \sigma_x^2 = \lim_{T\to\infty}\int_0^T (x(t)-\bar{x})^2 dt$
- $\frac{dR(\tau=0)}{d\tau} = 0$
- $R(\tau=\infty) = 0$
- If $x(t) = y_1(t) + y_2(t) + y_3(t) + ... + y_n(t)$ where the y 's are independent, then $R_x(\tau) = R_{y1}(\tau) + R_{y2}(\tau) + ... + R_{yn}(\tau)$ and since $R_x(\tau=0) = \sigma_x^2$, then $\sigma^2{}_x(\tau) = \sigma^2{}_{y1} + \sigma^2{}_{y2} + ... + \sigma^2{}_{yn}$

Power Spectrum

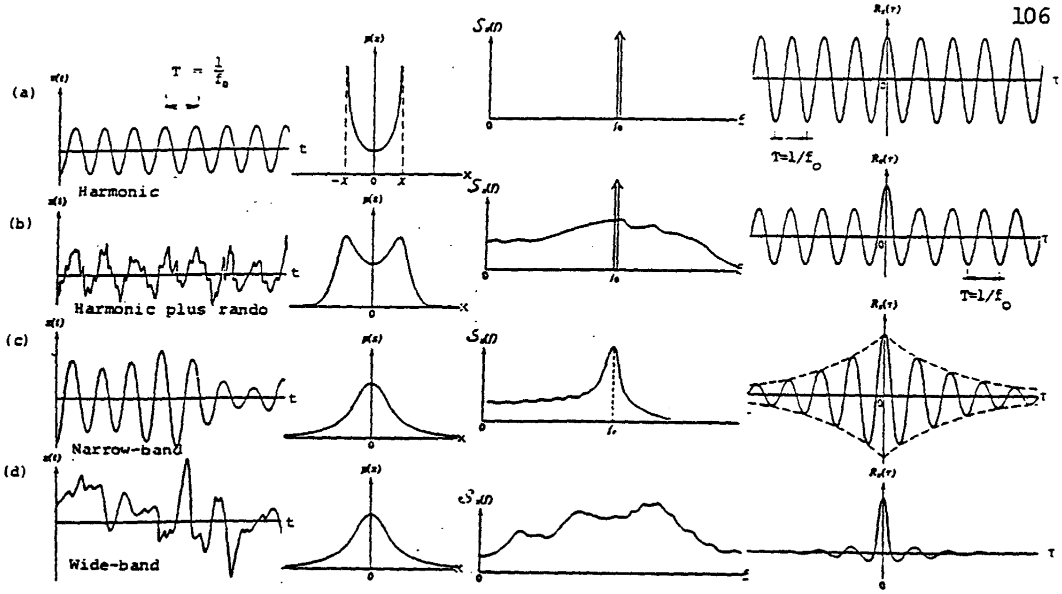
- Two-sided spectrum $G(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau$
- One-sided spectrum *S(f)* defined for $f \geq 0$

  $S(f) = 2G(f)$ and recalling that $R(\tau) = R(-\tau)$

  $$\left.\begin{aligned} S(f) &= 4\int_{0}^{\infty} R(\tau)\cos(2\pi f\tau)d\tau \\ R(\tau) &= \int_{0}^{\infty} S(f)\cos(2\pi f\tau)df \end{aligned}\right\} \text{This is a “Cosine Fourier Transform Pair”}$$

- The area under the power spectrum equals the variance, σ^2
- Dimensions of power spectrum are (dimension of x^2)/frequency, for example if *“x”* is displacement in metres, then the unsits of the power spectrum are: m^2 / Hz
- Power spectra are often normalized by the variance, σ^2, so the area is 1.0
- Examples of Random Processes:

x(t) *p(x)* *$S_X(f)$* *$R_X(\tau)$*



Power Spectra

We defined a Power Spectral Density Function:

$$G(f) = \int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau$$

$G(f)$ is a symmetric function about $f = 0$. Since $f < 0$ have little physical meaning, one defines a one-sided power spectrum as $S(F)$ for $f \geq 0$.

To conserve equality of variance;

$$S(f) = 2G(f) = 2\int_{-\infty}^{\infty} R(\tau) e^{-i2\pi f\tau} d\tau$$

$$= 2\int_{-\infty}^{\infty} R(\tau)\cos 2\pi f\tau d\tau + i2\int_{-\infty}^{\infty} R(\tau)\sin 2\pi f\tau d\tau$$

= 0 Integral of product of even & odd functions

$S(f) = 4\int_0^{\infty} R(\tau)\cos 2\pi f\tau d\tau$ … the one-sided SDF, a Fourier Transform

Properties of Power Spectra

- Relationship between the Probability Density Function (PDF) and the Spectral Density Function (SDF):

  If $x(t)$ is Gaussian;

  $$p_x(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$

  where

  $$\sigma_x^2 = \int_0^{\infty} S_x(f) df \qquad (8.18)$$

  When $y(t) = ax(t)$ … $S_y(f) = a^2 S_x(f)$ and $\sigma_y^2 = a^2\sigma_x^2$

  e.g. if $p(t) = -m\ddot{y}_g(t)$ then $S_p(f) = m^2 S_{\ddot{y}_g}(f)$

  If $x(t)$ is a time-varying quantity, then

  $p_x(x)$ … amplitude domain
  $S_x(f)$ … frequency domain
  $R_x(\tau)$ … correlation with itself at time lag τ *(see page 106)*

How do we describe the time variation of a periodic function?

using a Fourier Series: provides information on variation in frequency at ω, 2ω, 3ω etc. (at discrete harmonics)

How do we describe a Random Variable in Time?

using a Fourier Integral: provides information on variation at all values of ω

$$\left.\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)e^{i\omega t}d\omega \\ \\ A(\omega) &= \int_{-\infty}^{\infty} x(t))e^{-i\omega t}dt \end{aligned}\right\}\text{a Fourier Transform Pair}$$

Response of Structures to Random Loads

If $P(t) = P_o e^{i\omega t} = P_o e^{i2\pi ft}$ (a single harmonic), then the response of the structure becomes:

$$y(t) = \frac{P_o}{K}\underset{\uparrow}{H(f)}e^{i2\pi ft}$$

The Complex Mechanical Admittance: $H(f) = \dfrac{1}{1-\left(\dfrac{f}{f_o}\right)^2 + i2D\dfrac{f}{f_o}}$ *(see pg 29b)*

If the load consists of many harmonics;

$$y(t) = \sum_{-\infty}^{\infty}\frac{1}{k}c_r H(f_r)e^{ir2\pi f_1 t}$$

$$\overline{y^2} = \sigma_y^2 = \frac{1}{k^2}\sum_{-\infty}^{\infty}\left|c_r\right|^2\left|H(f_r)\right|^2$$

If we represent a random load by a periodic function with $T \to \infty$ (i.e it never repeats itself)

As $1/T \to df$, $rf_1 = f_r \to f$, and $\sum_0^{\infty} \to \int_0^{\infty}$

So, $\overline{y^2} = \int_0^{\infty} S_y(f)df = \int_0^{\infty} S_y(f)df = \dfrac{1}{k^2}\int_0^{\infty} S_p(f)\underset{\uparrow}{\left|H(f)\right|^2}df$

the Modulus of the Complex Mechanical Admittance

$$\left|H(f)\right|^2 = \varepsilon^2 = \frac{1}{\left[1-(f/f_o)^2\right]^2 + 4D^2(f/f_o)^2}$$ *(see pg 111)*

Variance of Response to Random Load *P(t)*

$$\overline{y^2} = \int_0^\infty S_y(f)df$$

$$= \frac{1}{K^2}\int_0^\infty S_p(f)\left|H(f)\right|^2 df$$

Difficult to evaluate in closed form

A conservative approximation is:

$$\overline{y^2} \cong \frac{1}{k^2}\int_o^\infty S_p(f)df + \frac{1}{k^2}\int_0^\infty S_p(f_o)\left|H(f)\right|^2 df$$

$$= \frac{1}{k^2}\left[\sigma_p^2 + \frac{1}{k^2}S_p(f_o)\frac{\pi f_o}{4D}\right]$$

$$\sigma_y^2 = \sigma_{yB}^2 + \sigma_{yR}^2$$

Resonant

Background

NOTE:

$$\sigma_{yR}^2 \cong \frac{\pi}{4}\frac{f_o}{D}S_P(f_o)$$

$$\sigma_{yR} \cong \sqrt{\frac{\pi}{4}\frac{f_o}{D}S_P(f_o)} \quad \propto \quad \frac{1}{\sqrt{D}}$$

random

$$\propto \quad \frac{1}{D}$$

harmonic

Peak Value of a Random Process (*pg 114)*

$$\hat{y} = \overline{y} + g\sigma_y \quad \textit{(see fig 8.15)}$$

RMS value or standard deviation

Peak factor

Mean

Peak response

$$g = \sqrt{2\log_e \nu T} + \frac{0.5772}{\sqrt{2\log_e \nu T}}$$

$$\nu = \sqrt{\frac{\int_0^\infty f^2 S(f)df}{\int_0^\infty S(f)df}} \approx f_o \text{ for a narrow-band process}$$

The cycling rate, or apparent frequency

Response to Gusting Wind

Point Structure is one where $\sqrt{A} << \lambda(f)$

$$F_D(t) = \frac{1}{2}\rho C_D A V^2(t) = \frac{1}{2}\rho C_D A\left[\overline{V} + v(t)\right]^2 \text{ where } \overline{V} >> v(t) \text{ so,}$$

$$F_D(t) = \frac{1}{2}\rho C_D A\overline{V}^2 + \rho C_D A\overline{V}v(t) \quad \text{a linearization of the response}$$

mean time varying

$$S_D(f) = \frac{4\overline{F}_D^2}{\overline{V}^2} S_v(f)$$

BUT Real structures have $\sqrt{A} \approx \lambda(f)$

Therefore the instantaneous wind induced pressures are not fully correlated over the frontal area of the body, so:

$$S_F = \frac{4\overline{F}_D^2}{\overline{V}}\left|\chi\left(\frac{fL}{\overline{V}}\right)\right|^2 S_v(f)$$

The Aerodynamic Admittance

$$\left|\chi\left(\frac{\sqrt{A}}{\lambda_f}\right)\right|^2 \to 1 \text{ as } \frac{\sqrt{A}}{\lambda_f} \to 0 \text{ and } \left|\chi\left(\frac{\sqrt{A}}{\lambda_f}\right)\right|^2 \to 0 \text{ as } \frac{\sqrt{A}}{\lambda_f} \to \infty$$

The National Building Code of Canada (NBCC)

Uses a Gust Factor approach

$$\hat{y} = C_g \overline{y}$$

mean

gust effect factor

peak

The NBCC contains a procedure for the evaluation of C_g

For:

i) simple shape buildings
ii) 1^{st} sway mode
iii) mass distribution must be constant with height
iv) only drag is considered (lift or crosswind response is not predicted)