REVIEW OF SDOF CONCEPTS

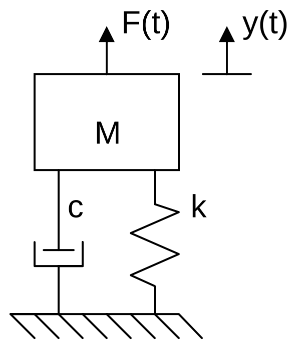


Fig. 1 SDOF Oscillator

Equation of motion describing "dynamic" equilibrium:

$$M\ddot{y} + c\dot{y} + ky = F(t) \qquad (1)$$

it is useful to examine several limiting cases:

A) FREE UNDAMPED VIBRATIONS (HOMOGENEOUS EQUATION)

i.e. F(t) = 0 and c=0

$$M\ddot{y} + ky = 0 \qquad (2)$$

and assuming the motion of the mass can be described by the sum of sinusoidal components:

$$y(t) = C_1 \sin\omega_o t + C_2 \cos\omega_o t \qquad (3)$$

where $\omega_o = \sqrt{\dfrac{k}{M}}$ the "undamped" natural frequency

Constants C_1 and C_2 are determined from initial conditions (e.g. if the mass is held at a given initial displacement and released, $C_1 = 0$ and $C_2 = y_o$)

the motion can also be written in terms of an initial displacement and a phase-shift:

$$y(t) = y_o \sin(\omega_o t + \phi) \qquad (4)$$

where the initial displacement y_o, is:

$$y_o = \sqrt{(C_1^2 + C_2^2)} \tag{5a}$$

and the phase-shift ϕ, is:

$$\phi = \tan^{-1}(C_2 / C_1) \tag{5b}$$

B) FREE DAMPED VIBRATIONS (HOMOGENEOUS EQUATIONS)

$$M\ddot{y} + c\dot{y} + ky = 0 \qquad \text{divide by } M$$

$$\ddot{y} + \frac{c}{M}\dot{y} + \frac{k}{M}y = 0 \tag{6}$$

and introducing; $\omega_o^2 = k / M$ and $2\alpha = c / M$

$$\ddot{y} + 2\alpha\dot{y} + \omega_o^2 y = 0 \tag{7}$$

a Particular Solution to this differential equation is:

$$y = Ce^{pt} \text{ so } \dot{y} = Cpe^{pt} \text{ and } \ddot{y} = Cp^2 e^{pt}$$

substituting, we now get a quadratic equation in the variable, *p*:

$$\begin{aligned} &p^2 + 2\alpha p + \omega^2 = 0 \\ &p_{1,2} = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}}{2} \\ &p_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2} \end{aligned} \tag{8abc}$$

defining the damping ratio as: $D = \alpha / \omega = \varsigma$, then:

$$p_{1,2} = -\alpha \pm \omega_o \sqrt{D^2 - 1} \tag{9}$$

therefore the solution (magnitude of the motion), depends on the magnitude of the damping *D*, relative to 1. There are 3 possible situations:

D is less than 1 (termed subcritical damping)
D is equal to 1 (termed critical damping), or
D is greater than 1 (termed supercritical damping)

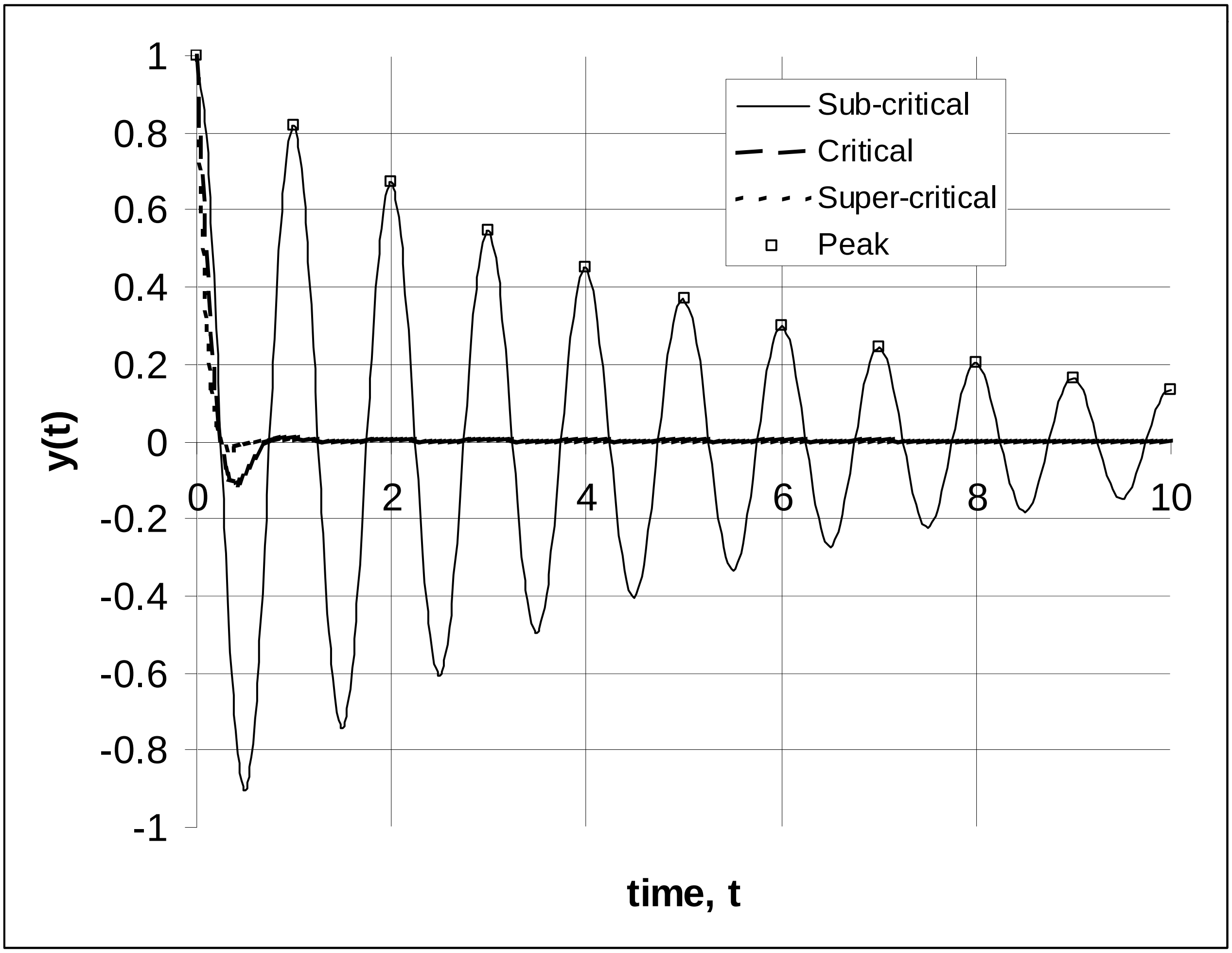


Fig. 2. **Free Damped Oscillations (*f* = 1.0 Hz, *D* = 0.032, critical and supercritical)**

Only the Subcritical case with $D < 1$, results in oscilliatory response. The other two cases are not relevant to civil engineering.

<u>For D < 1</u>

The solution of free vibrations becomes:

$$y(t) = y_o e^{-\alpha t} \sin(\omega_o' t + \phi_o) \qquad (10)$$

where $\omega_o' = \omega_o \sqrt{1 - D^2}$, the <u>damped</u> natural frequency
(recall that the <u>undamped</u> natural frequency, $\omega_o = \sqrt{k / M}$)

For most civil engineering applications, the structural damping is <u>much</u> less than 1, typically of the order of 0.001 (for a bridge cable) to 0.05 (for cracked concrete) so for practical purposes, $\omega_o = \omega_o'$

There are two important cases that arise with free damped vibrations:

i) when the initial velocity is zero, but with a finite initial amplitude
ii) when the initial amplitude is zero, but with a finite initial velocity

i) $y_o = A$ and $\dot{y}_o = 0$

$$y(t) = Ae^{-\alpha t}\cos\omega_o' t \approx Ae^{-\alpha t}\cos\omega_o t \qquad (11)$$

The structural damping of a system (e.g. chimney, flagpole, bridge) can be estimated by utilizing this relationship. Often, a structure can be given an initial displacement, *A,* then released and the subsequent decay with amplitude recorded.

The decrease in amplitude between successive cycles of vibration is dependent on the damping (recall that $D = \alpha / \omega$). The amplitude *y(t)* for multiples of the period of vibration N*$(2\pi / \omega_o)$ at the same point in the oscillation, will yield a straight line when A_{N+1} / A is plotted vs. N

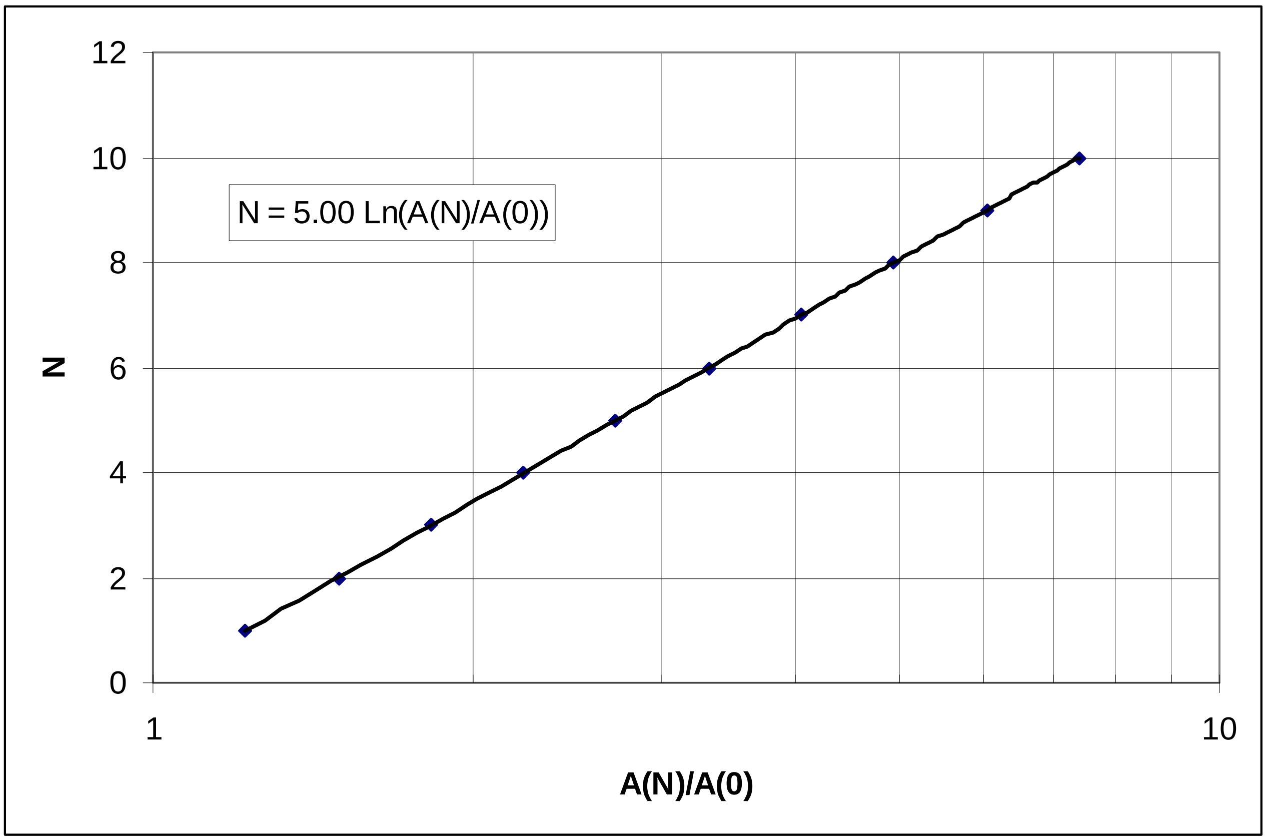


Fig. 3. Fit to Decay in Peak Amplitude Ratio vs. Cycle Number

From the above fit, the structural damping would be: $\delta = -\ln\left(\frac{A_{N+1}}{A_N}\right) \cong 2\pi D = \frac{1}{5} = 0.2$

(logarithmic decrement) or $D = \frac{0.2}{2\pi} = 0.032 = 3.2\%$

ii) $y_o = 0$ and $\dot{y}_o = V$

$$y(t) = \frac{V}{\omega_o'} e^{-\alpha t} \sin \omega_o' t \cong \frac{V}{\omega_o} e^{-\alpha t} \sin \omega_o t \qquad (12)$$

An example of this type of situation would be with a body initially at rest and experience an impact with another body.

C) RESPONSE TO HARMONIC EXCITATION

$$M\ddot{y} + c\dot{y} + ky = F_o \cos \omega t \qquad (13)$$

The steady-state response (once the influence of the initial conditions is lost) is:

$$y(t) = \frac{F_o}{K} \eta \cos(\omega t + \phi) \qquad (14)$$

where $\frac{F_o}{K} = y_{st}$, the static deflection under force, F_o

$$\eta = \frac{1}{\left[\left(1 - \frac{\omega^2}{\omega_o^2}\right)^2 + 4D^2 \frac{\omega^2}{\omega_o^2} \right]^{1/2}}$$

, the dynamic amplification factor, or the mechanical admittance (15a)

$$\phi = \tan^{-1}\left[\frac{2D\omega / \omega_o}{(1 - \omega / \omega_o)^2} \right]$$

, the phase angle between the excitation and the response (15b)

The complete solution is the summation of the steady-state response and the solution of the "Homogeneous" equation ($M\ddot{y} + c\dot{y} + ky = 0$), when evaluated with specific initial conditions.

$$y(t) = y_{st} \eta \cos(\omega t + \phi) + y_o e^{-\alpha t} \sin(\omega_o' t + \phi_o) \qquad (16)$$

y(t) = Steady-state Solution + Transient Solution

the transient solution "damps" out or disappears when the initial disturbance is past.

STEADY-STATE RESONANT VIBRATIONS

At resonance, the excitation frequency, ω is equal to the natural frequency, ω_o, and the dynamic amplification factor becomes:

$$\eta = \left(\frac{1}{4D^2}\right)^{1/2} = \frac{1}{2D} \tag{17a}$$

The phase angle becomes:

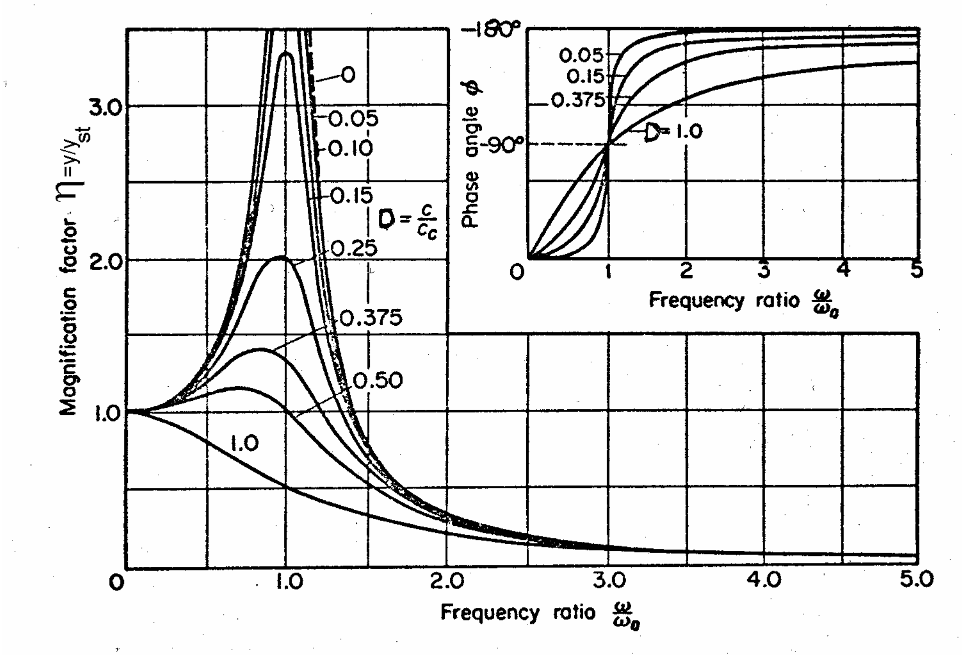
$$\phi = \tan^{-1}\left[\frac{2D}{0}\right] = -90^o = -\frac{\pi}{2} \tag{17b}$$

The response becomes:

$$y(t) = y_{st}\left(\frac{1}{2D}\right)\cos(\omega t - \frac{\pi}{2}) \tag{18}$$

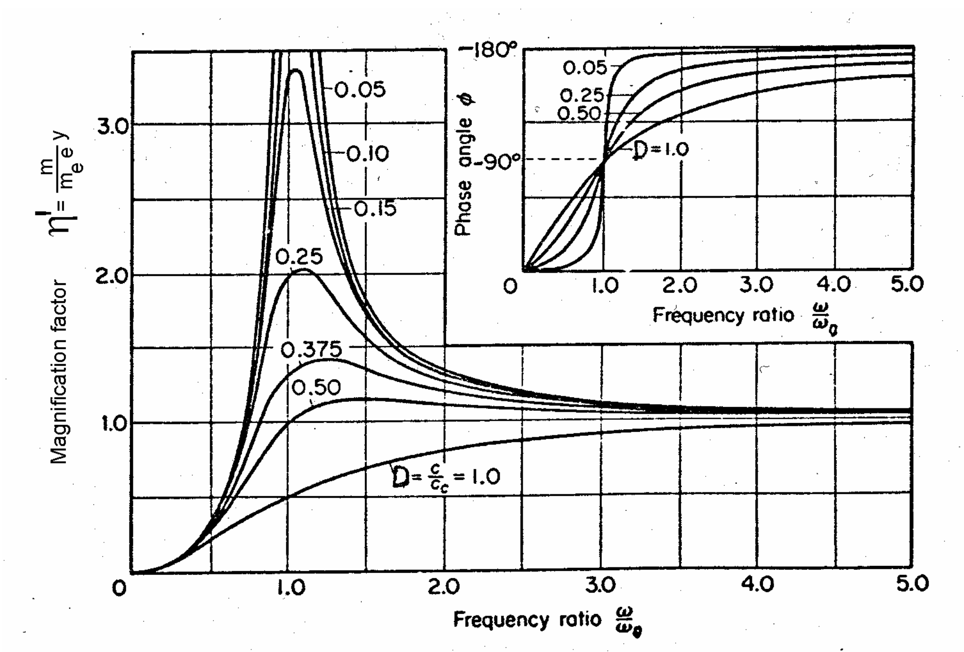
The maximum response is:

$$y(t)_{\max} = y_{st}\,\frac{1}{2D} \tag{19}$$



a) Constant Force Excitation

$$P(t) = P_o \cos \omega t \text{ and } y(t) = \frac{P_o}{k} \eta \cos(\omega t + \phi)$$



b) Quadratic Excitation, where $\eta' = \eta \frac{\omega^2}{\omega_o^2} = \frac{y}{m_e e / m}$

$$P(t) = m_e e \omega^2 \cos \omega t \text{ and } y(t) = \frac{m_e e \omega^2}{k} \eta \cos(\omega t + \phi) = \frac{m_e e}{m} \frac{\omega^2}{\omega_o{}^2} \eta \cos(\omega t + \phi)$$

Fig. 4. Dimensionless response to Harmonic Loads (Dynamic Amplitudes and Phase Shift)