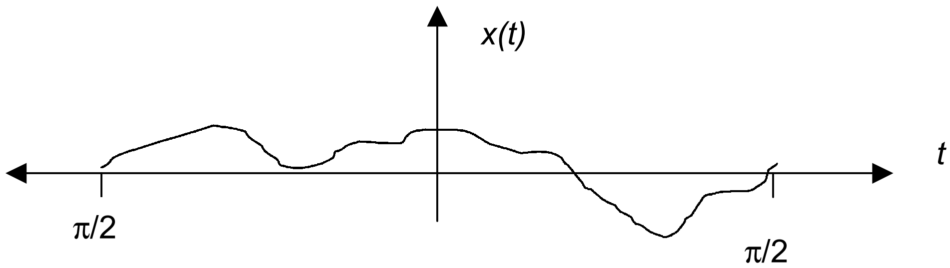
Time Series Analysis Concepts

Any periodic function *x(t)* can be represented by a Fourier Series



$$x(t) = a_o + \sum_{r=1}^{\infty} (a_r \cos r\omega_1 t + b_r \sin r\omega_1 t) \tag{1}$$

where

$\omega_1 = \frac{2\pi}{T}$; since $\omega_1 = 2\pi f_1$, then $f_1 = \frac{\omega_1}{2\pi} = \frac{1}{T}$

$T \equiv$ the period of *x(t)*

The constants a_o, a_r and b_r are evaluated as:

$$a_o = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt\,, \qquad \text{i.e the mean value} \tag{2a}$$

$$a_r = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos r\omega_1 dt \tag{2b}$$

$$b_r = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin r\omega_1 dt \tag{2c}$$

The mean square value of *x(t)* can also be expressed in terms of the coefficients a_o, a_r and b_r , squaring and averaging by the period *T.*

$$\sigma_x^2 = \overline{x^2(t)} = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)dt \tag{3}$$

substituting Eq. 1 in Eq. 3 we obtain

$$\sigma_x^2 = \overline{x^2(t)} = \frac{1}{T}\int_{-T/2}^{T/2} \left\{ a_o + \sum_{r=1}^{\infty} a_r \cos r\omega_1 t + \sum_{r=1}^{\infty} b_r \sin r\omega_1 t \right\}^2 dt \tag{4}$$

All terms involving a product of sin and cosine terms will be zero and the only ones left will be those containing a_o^2 and the terms involving either the product of cosines or sines, i.e. $(a_k \cos k\omega_1 t)(a_\ell \cos \ell\omega_1 t)$ or $(b_k \sin k\omega_1 t)(b_\ell \sin \ell\omega_1 t)$ and only when $k=\ell$. So the mean square value of *x(t)* becomes:

$$\begin{aligned}\sigma_x^2 = \overline{x^2(t)} &= \frac{1}{T}\left[a_o^2 T + \sum_{r=1}^{\infty}\left(a_r^2 \frac{T}{2} + b_r^2 \frac{T}{2}\right)\right] \\ &= a_o^2 + \frac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2)\end{aligned} \qquad (5)$$

Complex Fourier Series

We can alternatively express a periodic function using complex notation since:

$$e^{ir\omega_1 t} = \cos r\omega_1 t + i \sin r\omega_1 t \qquad (6)$$

Therefore:

$$x(t) = \sum_{r=-\infty}^{\infty} c_r e^{ir\omega_1 t} \qquad \text{(see Eq. 8.23)} \qquad (7)$$

where:

$$c_r = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-ir\omega_1 t}\, dt \qquad (8)$$

so the mean square amplitude (variance) becomes:

$$\overline{x^2(t)} = \frac{1}{T}\int_{-T/2}^{T/2}\left\{\sum_{-\infty}^{\infty} c_r e^{ir\omega_1 t}\right\}^2 dt \qquad (9)$$

This can be rewritten as:

$$\begin{aligned}\overline{x^2(t)} &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\int_{-T/2}^{T/2} c_n c_m e^{in\omega_1 t} e^{im\omega_1 t}\, dt \\ &= \sum_{-\infty}^{\infty} |c_r|^2\end{aligned} \qquad (10)$$

where $|c_r|^2 = c_r c_r^*$, the modulus of c_r, and c_r^* is the complex conjugate of c_r

Fourier Integrals and Fourier Transforms

We can extend the Fourier Series concept to a random or a non-periodic function by assuming that the period becomes very large. In the limit,

$$T \to \text{very large}$$

$$\omega = \frac{2\pi}{T} \to \text{very small}$$

$$= \Delta\omega$$

$$\text{so } \frac{1}{T} = \frac{\Delta\omega}{2\pi}$$

Now, we can express *x(t)* as:

$$x(t) = \sum_{-\infty}^{\infty} \underbrace{\left\{ \frac{\Delta\omega}{2\pi} \int_{-T/2}^{T/2} x(t) e^{-ir\Delta\omega t} dt \right\}}_{C_r} e^{ir\Delta\omega t} \quad (11)$$

Now we assume that $T \to \infty$, $\Delta\omega = d\omega$ and $r\Delta\omega \to \omega$. This allows a continuous description with ω, rather than in terms of discrete multiples or harmonics of ω_1. The summation now becomes an integration and *x(t)* is now described by a Fourier Integral:

$$x(t) = \int_{-\infty}^{\infty} \left[\frac{d\omega}{2\pi} \int x(t) e^{-i\omega t} dt \right] e^{i\omega t} \quad (12)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega) e^{i\omega t} d\omega \quad (13)$$

where

$$A(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \quad (14)$$

$x(t)$ and $A(\omega)$ are referred to as a *Fourier Transform Pair*

Recall that we have introduced $R(\tau)$ and $G(\tau)$ as a *Fourier Transform Pair* and $R(\tau)$ and $S(f)$ as a *Cosine Transform Pair*