Solutions to Assignment #11

1)

16. Let S_1 be the lateral surface, S_2 the top disk, and S_3 the bottom disk.

On S_1: $\mathbf{r}(\theta, z) = 3\cos\theta\,\mathbf{i} + 3\sin\theta\,\mathbf{j} + z\,\mathbf{k}$, $0 \le \theta \le 2\pi$, $0 \le z \le 2$, $|\mathbf{r}_\theta \times \mathbf{r}_z| = 3$.

$\iint_{S_1}(x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^2 (9 + z^2)\,3\,dz\,d\theta = 2\pi(54 + 8) = 124\pi$.

On S_2: $\mathbf{r}(\theta, r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + 2\,\mathbf{k}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$.

$\iint_{S_2}(x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 - 4)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4} + 18\right) = \frac{153}{2}\pi$.

On S_3: $\mathbf{r}(\theta, r) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$, $|\mathbf{r}_\theta \times \mathbf{r}_r| = r$.

$\iint_{S_3}(x^2 + y^2 + z^2)\,dS = \int_0^{2\pi}\int_0^3 (r^2 + 0)\,r\,dr\,d\theta = 2\pi\left(\frac{81}{4}\right) = \frac{81}{2}\pi$.

Hence $\iint_S (x^2 + y^2 + z^2)\,dS = 124\pi + \frac{153}{2}\pi + \frac{81}{2}\pi = 241\pi$.

2)

40. $\rho(x, y, z) = 1500$, $\mathbf{F} = \rho\mathbf{V} = (1500)(-y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k})$. S is given by

$\mathbf{r}(\phi, \theta) = 5\sin\phi\cos\theta\,\mathbf{i} + 5\sin\phi\sin\theta\,\mathbf{j} + 5\cos\phi\,\mathbf{k}$, $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$, and

$\mathbf{r}_\phi \times \mathbf{r}_\theta = 25\sin^2\phi\cos\theta\,\mathbf{i} + 25\sin^2\phi\sin\theta\,\mathbf{j} + 25\sin\phi\cos\phi\,\mathbf{k}$. Thus the rate of outward flow is

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = 1500\int_0^{2\pi}\int_0^{\pi}\left(-125\sin^3\phi\sin\theta\cos\theta + 125\sin^3\phi\sin\theta\cos\theta + 250\sin\phi\cos^2\phi\right)d\phi\,d\theta$$

$$= (3000\pi)(250)\left(-\tfrac{1}{3}\cos^3\phi\right)\Big]_0^{\pi} = 500{,}000\pi.$$

3)

6. The boundary curve C is the unit circle in the yz-plane. By Equation 3,

$\iint_{S_1}\text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2}\text{curl}\,\mathbf{F}\cdot d\mathbf{S}$ where S_1 is the original hemisphere and S_2 is the disk

$y^2 + z^2 \le 1$, $x = 0$. curl $\mathbf{F} = (x - x^2)\,\mathbf{i} - (y + e^{xy}\sin z)\,\mathbf{j} + (2xz - xe^{xy}\cos z)\,\mathbf{k}$, and for S_2 we choose $\mathbf{n} = \mathbf{i}$

so that C has the same orientation for both surfaces. Then curl $\mathbf{F}\cdot\mathbf{n} = x - x^2$ on S_2, where $x = 0$. Thus

$\iint_{S_2}\text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \iint_{y^2+z^2\le 1}(x - x^2)\,dA = \iint_{y^2+z^2\le 1} 0\,dA = 0.$

Alternatively, we can evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$: C with positive orientation is given by $\mathbf{r}(t) = \langle 0, \cos t, \sin t\rangle$, $0 \le t \le 2\pi$, and

$$\iint_S \text{curl}\,\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_0^{2\pi}\left\langle e^{0(\cos t)}\cos(\sin t), (0)^2(\sin t), (0)(\cos t)\right\rangle\cdot\langle 0, -\sin t, \cos t\rangle\,dt$$

$$= \int_0^{2\pi} 0\,dt = 0$$