Solutions to Assignment #4

1)

12. $\rho(x,y) = k(x^2+y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\,dr\,d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{5}k\left[\sin\theta\right]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{5}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{5}k$.

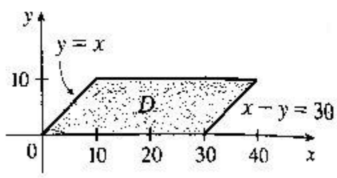
Hence $(\overline{x},\overline{y}) = \left(\frac{8}{5\pi},\frac{8}{5\pi}\right)$.

2)

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x)f_2(y) = \begin{cases} \frac{1}{50}e^{-x}y & \text{if } x \geq 0,\ 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \leq 30$. Thus the probability that they meet is $P((X,Y)\in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



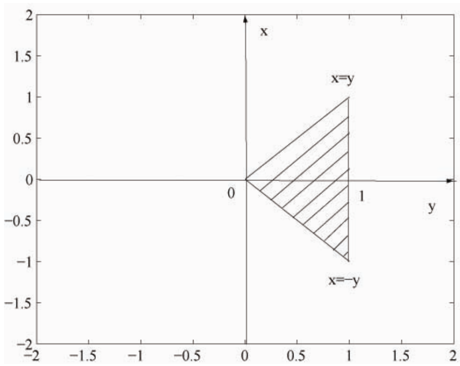
$$P((X,Y)\in D) = \iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30}\frac{1}{50}e^{-x}y\,dx\,dy = \frac{1}{50}\int_0^{10}y\left[-e^{-x}\right]_{x=y}^{x=y+30}dy$$

$$= \frac{1}{50}\int_0^{10}y(-e^{-(y+30)}+e^{-y})\,dy = \frac{1}{50}(1-e^{-30})\int_0^{10}ye^{-y}\,dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is

$\frac{1}{50}(1-e^{-30})\left[-(y+1)e^{-y}\right]_0^{10} = \frac{1}{50}(1-e^{-30})(1-11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance will meet. Such is student life!

3) Method (1)



$$y^2 + z^2 = 1 \Rightarrow z = \sqrt{1-y^2}$$

$$A = 8\int_0^1\int_{-y}^{y}\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}\,dxdy$$

$$= 8\int_0^1\int_{-y}^{y}\sqrt{1+0+(\frac{-2y}{2\sqrt{1-y^2}})^2}\,dxdy$$

$$= 16\int_0^1\int_0^{y}\sqrt{1+\frac{y^2}{1-y^2}}\;dxdy$$

$$= 16\int_0^1\int_0^{y}\frac{1}{\sqrt{1-y^2}}dxdy$$

$$= 16\int_0^1\frac{1}{\sqrt{1-y^2}}\int_0^{y}\;dxdy$$

$$= 16\int_0^1\frac{ydy}{\sqrt{1-y^2}}$$

$$= 16[-\sqrt{1-y^2}\,]|_0^1$$

$$= 16$$

Method (2)

24. First we find the area of the face of the surface that intersects the positive y-axis. As in Exercise 23, we can project the face onto the xz-plane, so the surface lies "above" the disk $x^2 + z^2 \le 1$. Then $y = f(x, z) = \sqrt{1 - z^2}$ and the area is

$$A(S) = \iint_{x^2+z^2\le 1} \sqrt{[f_x(x,z)]^2 + [f_z(x,z)]^2 + 1}\, dA = \iint_{x^2+z^2\le 1} \sqrt{0 + \left(\frac{-z}{\sqrt{1-z^2}}\right)^2 + 1}\, dA$$

$$= \iint_{x^2+z^2\le 1} \sqrt{\frac{z^2}{1-z^2} + 1}\, dA = \int_{-1}^{1} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\, dz$$

$$= 4 \int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\, dz \qquad \text{[by the symmetry of the surface]}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t\to 1^-} 4 \int_0^t \int_0^{\sqrt{1-z^2}} \frac{1}{\sqrt{1-z^2}}\, dx\, dz = \lim_{t\to 1^-} 4 \int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\, dz = \lim_{t\to 1^-} 4 \int_0^t dz = \lim_{t\to 1^-} 4t = 4.$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.