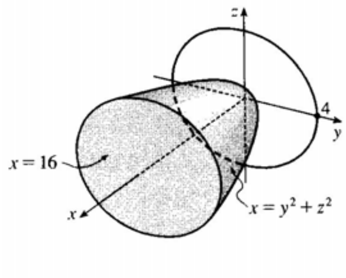
Solutions to Assignment #5

1)

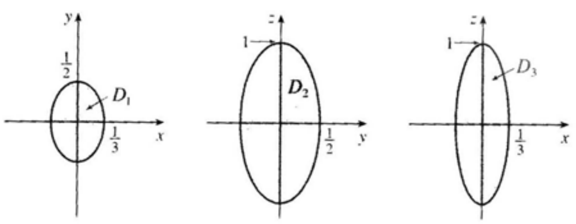
20. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$.
Thus, $E = \{(x, y, z) \mid y^2 + z^2 \le x \le 16, y^2 + z^2 \le 16\}$.
Let $D = \{(y, z) \mid y^2 + z^2 \le 16\}$. Then using polar coordinates
$y = r\cos\theta$ and $z = r\sin\theta$, we have

$$V = \iint_D \left(\int_{y^2+z^2}^{16} dx\right) dA = \iint_D (16 - (y^2 + z^2))\, dA$$
$$= \int_0^{2\pi} \int_0^4 (16 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 (16r - r^3)\, dr$$
$$= \left[\theta\right]_0^{2\pi} \left[8r^2 - \tfrac{1}{4}r^4\right]_0^4 = 2\pi(128 - 64) = 128\pi$$



2)

1.



If D_1, D_2 and D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \{(x, y) \mid 9x^2 + 4y^2 \le 1\}$, $D_2 = \{(y, z) \mid 4y^2 + z^2 \le 1\}$, $D_3 = \{(x, z) \mid 9x^2 + z^2 \le 1\}$. Therefore

$$\iiint_E f(x, y, z)\, dV = \int_{-1/3}^{1/3} \int_{-\sqrt{1-9x^2}/2}^{\sqrt{1-9x^2}/2} \int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x, y, z)\, dz\, dy\, dx$$
$$= \int_{-1/2}^{1/2} \int_{-\sqrt{1-4y^2}/3}^{\sqrt{1-4y^2}/3} \int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x, y, z)\, dz\, dx\, dy$$
$$= \int_{-1/2}^{1/2} \int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x, y, z)\, dx\, dz\, dy$$
$$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}/2}^{\sqrt{1-z^2}/2} \int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x, y, z)\, dx\, dy\, dz$$
$$= \int_{-1/3}^{1/3} \int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x, y, z)\, dy\, dz\, dx$$
$$= \int_{-1}^{1} \int_{-\sqrt{1-z^2}/3}^{\sqrt{1-z^2}/3} \int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x, y, z)\, dy\, dx\, dz$$

3)

24. In spherical coordinates, the sphere $x^2 + y^2 + z^2 = 4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2 + y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\{(\rho, \theta, \phi) \mid 0 \le \rho \le 2, 0 \le \theta \le 2\pi, \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\}$ and

$$V = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \int_0^{2\pi} d\theta \int_0^2 \rho^2 \, d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2} \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\,\pi}{3}$$