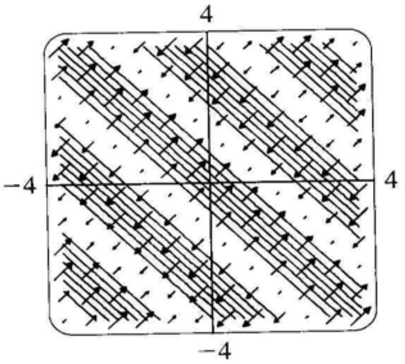
Solutions to Assignment #7

1)

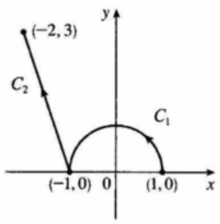
28. We graph ∇f along with a contour map of f.



The graph shows that the gradient vectors are perpendicular to the level curves. Also, the gradient vectors point in the direction in which f is increasing and are longer where the level curves are closer together.

2)

8.



$C = C_1 + C_2$

On C_1: $x = \cos t \;\Rightarrow\; dx = -\sin t\,dt,\ y = \sin t \;\Rightarrow\; dy = \cos t\ dt,\ 0 \le t \le \pi.$

On C_2: $x = -1 - t \;\Rightarrow\; dx = -dt,\ y = 3t \;\Rightarrow\; dy = 3\,dt,\ 0 \le t \le 1.$

Then

$$\begin{aligned}
\int_C \sin x\,dx + \cos y\,dy &= \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy \\
&= \int_0^{\pi} \sin(\cos t)(-\sin t\,dt) + \cos(\sin t)\cos t\,dt \\
&\qquad + \int_0^1 \sin(-1-t)(-dt) + \cos(3t)(3\,dt) \\
&= \big[-\cos(\cos t) + \sin(\sin t)\big]_0^{\pi} + \big[-\cos(-1-t) + \sin(3t)\big]_0^1 \\
&= -\cos(\cos\pi) + \sin(\sin\pi) + \cos(\cos 0) - \sin(\sin 0) \\
&\qquad - \cos(-2) + \sin(3) + \cos(-1) - \sin(0) \\
&= -\cos(-1) + \sin 0 + \cos(1) - \sin 0 - \cos(-2) + \sin 3 + \cos(-1) \\
&= -\cos 1 + \cos 1 - \cos 2 + \sin 3 + \cos 1 = \cos 1 - \cos 2 + \sin 3
\end{aligned}$$

where we have used the identity $\cos(-\theta) = \cos\theta$.

3)

32. We use the parametrization $x = r\cos t$, $y = r\sin t$, $0 \le t \le \frac{\pi}{2}$.

Then $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt = r\,dt$, so

$m = \int_C (x+y)\,ds = \int_0^{\pi/2} (r\cos t + r\sin t)\,r\,dt = r^2\left[\sin t - \cos t\right]_0^{\pi/2} = 2r^2$,

$$\overline{x} = \frac{1}{2r^2}\int_C x(x+y)\,ds = \frac{1}{2r^2}\int_0^{\pi/2}(r^2\cos^2 t + r^2\cos t\sin t)r\,dt = \frac{r}{2}\left[\frac{t}{2} + \frac{\sin 2t}{4} - \frac{\cos 2t}{4}\right]_0^{\pi/2}$$

$$= \frac{r\,(\pi+2)}{8}, \text{ and}$$

$$\overline{y} = \frac{1}{2r^2}\int_C y(x+y)\,ds = \frac{1}{2r^2}\int_0^{\pi/2}(r^2\sin t\cos t + r^2\sin^2 t)r\,dt$$

$$= \frac{r}{2}\left[-\frac{\cos 2t}{4} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi/2} = \frac{r(\pi+2)}{8}.$$

Therefore $(\overline{x}, \overline{y}) = \left(\frac{r(\pi+2)}{8}, \frac{r(\pi+2)}{8}\right)$.