Solutions to Assignment #8

1)

10. $\dfrac{\partial(xy\cosh xy + \sinh xy)}{\partial y} = x^2y\sinh xy + x\cosh xy + x\cosh xy = x^2y\sinh xy + 2x\cosh xy = \dfrac{\partial(x^2\cosh xy)}{\partial x}$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$. Thus $\mathbf{F}$ is conservative, so there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y) = xy\cosh xy + \sinh xy$ implies $f(x,y) = x\sinh xy + g(y) \Rightarrow f_y(x,y) = x^2\cosh xy + g'(y)$. But $f_y(x,y) = x^2\cosh xy$ so $g(y) = K$ and $f(x,y) = x\sinh xy + K$ is a potential function for $\mathbf{F}$.

2)

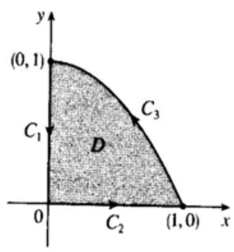
20. Here $\mathbf{F}(x,y) = (1 - ye^{-x})\,\mathbf{i} + e^{-x}\,\mathbf{j}$. Then $f(x,y) = x + ye^{-x}$ is a potential function for $\mathbf{F}$, that is, $\nabla f = \mathbf{F}$ so $\mathbf{F}$ is conservative and thus its line integral is independent of path. Hence $\int_C (1 - ye^{-x})\,dx + e^{-x}\,dy = \int_C \mathbf{F}\cdot d\mathbf{r} = f(1,2) - f(0,1) = (1 + 2e^{-1}) - 1 = 2/e$.

3)

4. (a) $C_1 : x = 0 \;\Rightarrow\; dx = 0\,dt,\ y = 1 - t \;\Rightarrow$
$dy = -dt,\ 0 \le t \le 1$

$C_2 : x = t \;\Rightarrow\; dx = dt,\ y = 0 \;\Rightarrow\; dy = 0\,dt,\ 0 \le t \le 1$

$C_3 : x = 1 - t \;\Rightarrow\; dx = -dt,\ y = 1 - (1-t)^2 = 2t - t^2 \;\Rightarrow$
$dy = (2 - 2t)\,dt,\ 0 \le t \le 1$



Thus

$$\begin{aligned}\oint_C x\,dx + y\,dy &= \oint_{C_1+C_2+C_3} x\,dx + y\,dy \\ &= \int_0^1 (0\,dt + (1-t)(-dt)) + \int_0^1 (t\,dt + 0\,dt) + \int_0^1 ((1-t)(-dt) + (2t - t^2)(2 - 2t)\,dt \\ &= \left[\tfrac{1}{2}t^2 - t\right]_0^1 + \left[\tfrac{1}{2}t^2\right]_0^1 + \left[\tfrac{1}{2}t^4 - 2t^3 + \tfrac{5}{2}t^2 - t\right]_0^1 \\ &= -\tfrac{1}{2} + \tfrac{1}{2} + \left(\tfrac{1}{2} - 2 + \tfrac{5}{2} - 1\right) = 0\end{aligned}$$

(b) $\oint_C x\,dx + y\,dy = \displaystyle\iint_D \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right] dA = \iint_D 0\,dA = 0$