Solutions to Assignment #9

1)

16. $\mathbf{F}(x,y) = \left\langle y - \ln(x^2+y^2), 2\tan^{-1}\left(\frac{y}{x}\right)\right\rangle$ and the region D enclosed by C is the disk with radius 1 centered at $(2,3)$. C is oriented positively, so

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C (y-\ln(x^2+y^2))\,dx + \left(2\tan^{-1}\left(\frac{y}{x}\right)\right)\,dy \\
&= \iint_D \left[\frac{\partial}{\partial x}\left(2\tan^{-1}\left(\frac{y}{x}\right)\right) - \frac{\partial}{\partial y}\left(y - \ln(x^2+y^2)\right)\right] dA \\
&= \iint_D \left[2\left(\frac{-yx^{-2}}{1+(y/x)^2}\right) - \left(1 - \frac{2y}{x^2+y^2}\right)\right] dA = \iint_D \left[-\frac{2y}{x^2+y^2} - 1 + \frac{2y}{x^2+y^2}\right] dA \\
&= -\iint_D dA = -(\text{area of } D) = -\pi
\end{aligned}$$

2)

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D(-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

3)

26.

$$\begin{aligned}
\operatorname{curl}(f\mathbf{F}) &= \left[\frac{\partial(fR_1)}{\partial y} - \frac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(fP_1)}{\partial z} - \frac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\frac{\partial(fQ_1)}{\partial x} - \frac{\partial(fP_1)}{\partial y}\right]\mathbf{k} \\
&= \left[f\frac{\partial R_1}{\partial y} + R_1\frac{\partial f}{\partial y} - f\frac{\partial Q_1}{\partial z} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\frac{\partial P_1}{\partial z} + P_1\frac{\partial f}{\partial z} - f\frac{\partial R_1}{\partial x} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} \\
&\quad + \left[f\frac{\partial Q_1}{\partial x} + Q_1\frac{\partial f}{\partial x} - f\frac{\partial P_1}{\partial y} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k} \\
&= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k} \\
&\quad + \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k} \\
&= f\operatorname{curl}\mathbf{F} + (\nabla f)\times\mathbf{F}
\end{aligned}$$