THE UNIVERSITY OF WESTERN ONTARIO

London Ontario

Calculus 251b

Midterm Test – February 18, 2007

2:00 – 5:00 p.m.

3 HOURS

Name: Solution **Stu. #:** ________ **Tut. Section:** ____

Notes:

1) There are **9** questions in total. The first **8** questions are worth 100 (full) marks. The last question is a **Bonus**.
2) Text book, Mathematics Handbook and calculators are NOT allowed.
2) Except for the multiple choice questions, you must show all the steps in your calculation on the page with the question. (Rough work can be done on the back of the previous page.)
4) Do not remove any pages from the exam book. Do not remove staples. There should be 11 pages in the booklet (including the cover page).

Circle your (lecture) section

001 – P. Yu

002 – C. Drapaca

FOR INSTRUCTOR'S USE ONLY

1	2	3	4	5	6	7	8	Bonus	Total
30	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**10**	**100**

[30] 1. For the following multiple choice questions, circle one (only one) answer for each question. (3 marks for each question. No penalty for guessing.)

(1) The average value of $f(x,y) = x\sqrt{y^2 - x^2}$ over the region $D = \{(x,y) | 0 \le y \le 1,\ 0 \le x \le y\}$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{1}{12}$

(2) For $E = \{(\rho,\theta,\phi) | \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta,\phi) \le \rho \le g_2(\theta,\phi)\}$, $\iiint_E f(x,y,z)\, dV$ equals

(A) $\int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\cos\phi\sin\theta,\ \rho\cos\phi\cos\theta,\ \rho\sin\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(B) $\int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(C) $\int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\,\rho\sin\phi\, d\rho\, d\phi\, d\theta$

(D) $\int_\alpha^\beta \int_c^d \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\sin\theta,\ \rho\sin\phi\cos\theta,\ \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

(3) Let $I = \iint_R \sin(2x+y)\, dA$, where $R = [0,\frac{\pi}{2}] \times [0,\frac{\pi}{2}]$. Then I equals

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{3}{2}$ (D) $\frac{\pi^2}{4}$

(4) A lamina occupies the region inside the circle $x^2 + y^2 = 1$ of the first quadrant. If the density at any point of the lamina is proportional to its distance from the origin, the center of mass of the lamina $(\bar{x},\bar{y})$ is given by

(A) $(\frac{3}{2\pi},\frac{3}{2\pi})$ (B) $(2,2)$ (C) $(\frac{3}{\pi},\frac{2}{\pi})$ (D) $(\frac{\pi}{4},\frac{\pi}{4})$

(5) Define D as the region bounded by $|x| + |y| \le 1$, and D_1 as the region bounded by the line $x + y = 1$, the x-axis and the y-axis. Then $\iint_D (1+x+y)\, dx\, dy$ equals

(A) $4\iint_{D_1}(1+x+y)\, dx\, dy$ (B) 0 (C) 1 (D) 2

(6) If $\iint_D f(x,y)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{a\cos\theta} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$,

then the region D is

(A) $x^2+y^2 \le a^2$ (B) $x^2+y^2 \le a^2,\ x \ge 0$

(C) $x^2+y^2 \le a\,x,\ a>0$ (D) $x^2+y^2 \le a\,x,\ a<0$

(7) Given $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi$, then $\int_0^{\infty} \sqrt{x}\,e^{-x}\,dx$ is equal to

(A) $\sqrt{\pi}$ (B) $\frac{1}{2}\pi$ (C) π (D) $\frac{1}{2}\sqrt{\pi}$

(8) $\int_0^a \int_x^{\sqrt{2ax-x^2}} f(x,y)\,dy\,dx$ equals

(A) $\int_0^a \int_y^{\sqrt{2ay-y^2}} f(x,y)\,dx\,dy$ (B) $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{y} f(x,y)\,dx\,dy$

(C) $\int_0^a \int_y^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$ (D) $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

(9) $\int_0^a \int_x^a e^{y^2}\,dy\,dx$ equals

(A) $\frac{1}{2}(e^{a^2}-1)$ (B) $\frac{\sqrt{\pi}}{2}a(e^{a^2}-e^a)$ (C) $\frac{1}{2}a(ae^{a^2}-\frac{1}{a})$ (D) $\frac{\sqrt{\pi}}{2}a(e^{a^2}-1)$

(10) Suppose $D=\{(x,\,y)|x^2+y^2 \le R^2\}$, then $\iint_D (\frac{x^2}{a^2}+\frac{y^2}{b^2})\,dx\,dy$ equals

(A) $\frac{\pi}{4}R^2(a^2+b^2)$ (B) $\pi R^2(\frac{1}{a^2}+\frac{1}{b^2})$ (C) $\frac{\pi}{4}R^4(\frac{1}{a^2}+\frac{1}{b^2})$ (D) $\frac{\pi}{2}R^2(\frac{1}{a^2}+\frac{1}{b^2})$

[10] **2.** **(a)** Use Riemann Sum (suppose the limit of the Riemann Sum exists) to write the definition of the definite double integral $\iint_R f(x,y)\,dA$, where $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$.

(b) For $a = c = 0,\ b = d = 1,\ f(x,y) = x\,y$, find $\iint_R f(x,y)\,dA$ by calculating the limit of the Riemann Sum.

(a) $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$$

Taking equal partition, and choosing (x_{ij}^*, y_{ij}^*) as the upper-right corner, then

$\Delta A = (\frac{b-a}{m})(\frac{d-c}{n})$, so

$$\iint_R f(x,y)\,dA = (b-a)(d-c) \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)$$

(b) $$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m} \frac{1}{m} x_i \sum_{j=1}^{n} \frac{1}{n} y_j$$

$$= \left(\lim_{m\to\infty} \sum_{i=1}^{m} \frac{1}{m} x_i\right)\left(\lim_{n\to\infty} \sum_{j=1}^{n} \frac{1}{n} y_j\right)$$

$$= \left(\lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \frac{i}{m}\right)\left(\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \frac{j}{n}\right)$$

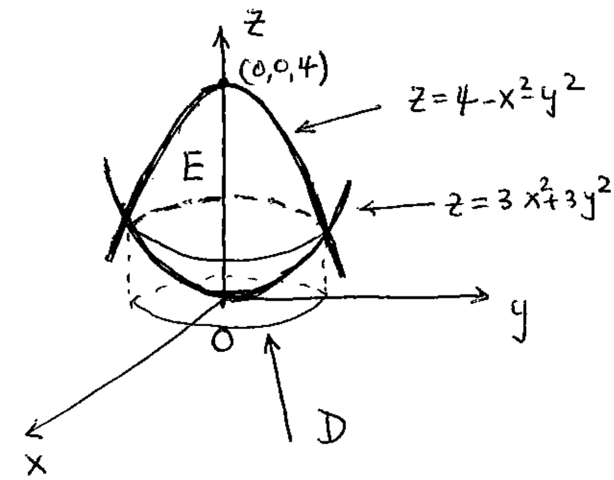
$$= \left(\lim_{m\to\infty} \frac{1}{m^2} \cdot \frac{1}{2} m(m+1)\right)^2$$

$$= \left(\frac{1}{2}\right)^2$$

$$= \frac{1}{4}$$

[10] **3.** Find the volume of the solid that is bounded by the paraboloids

$z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.



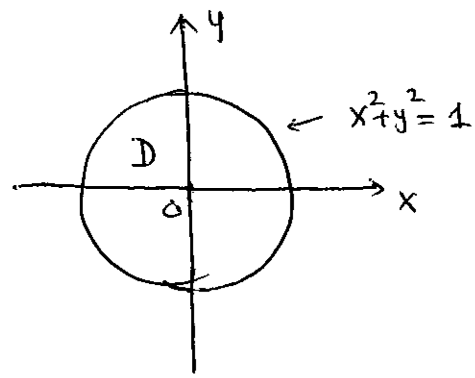
Projection of the intersection of the two paraboloid on the xy-plane:

$$\begin{cases} z = 3x^2+3y^2 \\ z = 4-x^2-y^2 \end{cases}$$

$\Rightarrow$

$$3x^2+3y^2 = 4-x^2-y^2$$

$\Rightarrow$ $x^2+y^2 = 1$



$$V = \iiint_E dV$$

$$= \iint_D \left[\int_{3x^2+3y^2}^{4-x^2-y^2} dz \right] dA \quad \text{(use polar coordinates)}$$

$$= \int_0^{2\pi} \int_0^1 \int_{3r^2}^{4-r^2} dz \; r\,dr\,d\theta$$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$$= 2\pi \int_0^1 (4-4r^2)\,r\,dr$$

$$= 2\pi \left[2r^2 - r^4 \right]_0^1$$

$$= 2\pi \;(\text{unit}^3)$$

[10] 4. Find the surface area of the part of the sphere $x^2+y^2+z^2=a^2$ that lies within the cylinder $x^2+y^2=a\,x$ and above the xy-plane.

$$A(s)=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

$$=\iint_D \frac{a}{\sqrt{a^2-x^2-y^2}}\,dA$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos\theta}\frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[-a\sqrt{a^2-r^2}\right]_{r=0}^{r=a\cos\theta}d\theta$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a\left(-a|\sin\theta|+a\right)d\theta$$

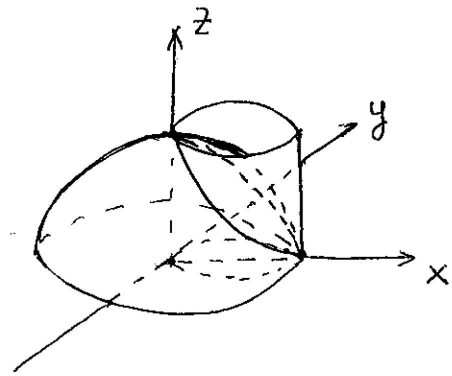
$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}a^2\left(1-|\sin\theta|\right)d\theta$$

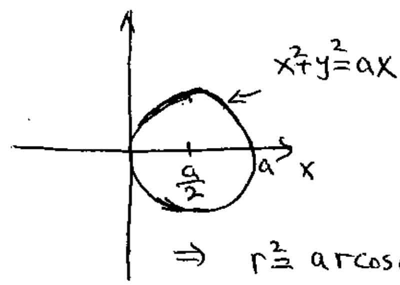
$$=4a^2\int_0^{\frac{\pi}{2}}(1-\sin\theta)\,d\theta$$

$$=4a^2\left[\theta+\cos\theta\right]_0^{\frac{\pi}{2}}$$

$$=4a^2\left(\frac{\pi}{2}+0-0-1\right)$$

$$=4a^2\left(\frac{\pi}{2}-1\right)\ (\text{unit}^2)$$





$\Rightarrow\ r^2=ar\cos\theta$

$r=a\cos\theta$

Polar coordinates.

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$$

$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$0\le r\le a\cos\theta$

$z=\sqrt{a^2-x^2-y^2}$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}$$

$$=\sqrt{1+\left(\frac{-2x}{2\sqrt{a^2-x^2-y^2}}\right)^2+\left(\frac{-2y}{2\sqrt{a^2-x^2-y^2}}\right)^2}$$

$$=\frac{a}{\sqrt{a^2-x^2-y^2}}$$

[10] **5.** **(a)** A lamp has two bulbs of a type with an average lifetime of 500 hours. Assuming that we can model the probability of failure of the bulbs by an exponential density function with mean $\mu = 500$, find the probability that both of the lamp's bulbs fail within 500 hours.

(b) Another lamp has just one bulb of the same type as in part (a). If one bulb burns out and is replaced by a bulb of the same type, find the probability that the two bulbs fail within a total of 500 hours.

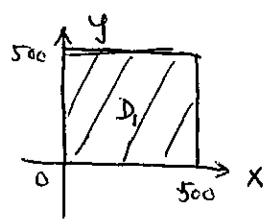
(**Hint:** Exponential density function is given by $\frac{1}{\mu} e^{-\frac{t}{\mu}}$ $\forall t \in [0, +\infty)$.)

$$f(x) = \begin{cases} \frac{1}{500} e^{-\frac{x}{500}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f(y) = \begin{cases} \frac{1}{500} e^{-\frac{y}{500}} & y \geq 0 \\ 0 & x < 0 \end{cases}$$

$$f(x,y) = f(x)f(y) = \begin{cases} \frac{1}{500^2} e^{-\frac{x}{500}} e^{-\frac{y}{500}}, & x \geq 0,\ y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) $P(0 \leq X \leq 500, 0 \leq Y \leq 500)$



$$= \iint_{D_1} f(x,y)\, dA$$

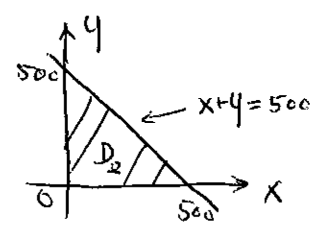
$$= \int_0^{500} \int_0^{500} \frac{1}{500^2} e^{-\frac{x}{500}} e^{-\frac{y}{500}}\, dx\, dy$$

$$= \int_0^{500} \frac{1}{500} e^{-\frac{x}{500}}\, dx \int_0^{500} \frac{1}{500} e^{-\frac{y}{500}}\, dy$$

$$= \left[-e^{-\frac{x}{500}}\right]_0^{500} \left[-e^{-\frac{y}{500}}\right]_0^{500}$$

$$= \left(1 - \frac{1}{e}\right)^2.$$

(b) $P(0 \leq X + Y \leq 500)$



$$= \iint_{D_2} f(x,y)\, dA$$

$$= \int_0^{500} \int_0^{500-x} \frac{1}{500^2} e^{-\frac{x}{500}} e^{-\frac{y}{500}}\, dy\, dx$$

$$= \int_0^{500} \frac{1}{500} e^{-\frac{x}{500}} \left[-e^{-\frac{y}{500}}\right]_{y=0}^{y=500-x} dx$$

$$= \int_0^{500} \frac{1}{500} e^{-\frac{x}{500}} \left[1 - e^{-\frac{500-x}{500}}\right] dx$$

$$= \int_0^{500} \frac{1}{500} \left(e^{-\frac{x}{500}} - \frac{1}{e}\right) dx$$

$$= \left[-e^{-\frac{x}{500}} - \frac{1}{500e} x\right]_0^{500} = -\frac{1}{e} - \frac{1}{e} + 1 = 1 - \frac{2}{e}.$$

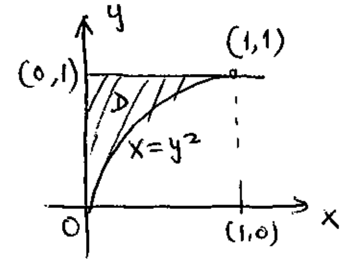
[10] **6. (a)** Find $\iint_D e^{\frac{x}{y}}\, dx\, dy$,

where D is the region bounded by $y^2 = x$, $x = 0$ and $y = 1$.

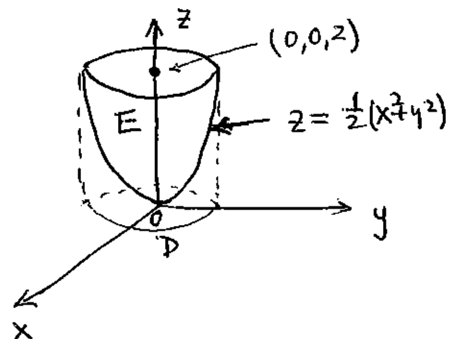
(b) Evaluate $\iiint_E x^2y^2z\, dV$,

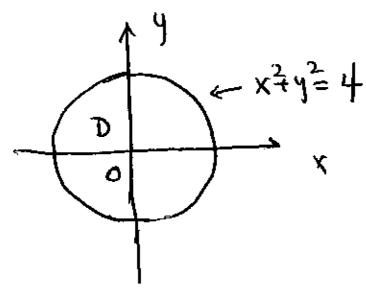
where E is the region bounded by the surface $2z = x^2 + y^2$ and the plane $z = 2$.

(a)
$$\iint_D e^{\frac{x}{y}}\, dx\, dy$$
$$= \int_0^1 \int_0^{y^2} e^{\frac{x}{y}}\, dx\, dy$$
$$= \int_0^1 \left[y e^{\frac{x}{y}} \right]_{x=0}^{x=y^2} dy$$
$$= \int_0^1 (y e^{y} - y)\, dy$$
$$= \left[y e^{y} - e^{y} - \tfrac{1}{2} y^2 \right]_0^1 = e - e - \tfrac{1}{2} + 1 = \tfrac{1}{2}.$$



(b)
$$\iiint_E x^2 y^2 z\, dV$$
$$= \iint_D \int_{\frac{x^2+y^2}{2}}^{2} x^2y^2z\, dz\, dA$$
$$= \int_0^{2\pi} \int_0^2 \int_{\frac{r^2}{2}}^{2} r^2\cos^2\theta\, r^2\sin^2\theta\, z\, dz\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} (\sin\theta\cos\theta)^2\, d\theta \int_0^2 \int_{\frac{r^2}{2}}^{2} z\, dz\, r^5\, dr$$
$$= \int_0^{2\pi} \tfrac{1}{4} (\sin 2\theta)^2\, d\theta \int_0^2 \left[\tfrac{1}{2} z^2\right]_{z=\frac{r^2}{2}}^{2} r^5\, dr$$
$$= \int_0^{2\pi} \tfrac{1}{8}(1 - \cos 4\theta)\, d\theta \int_0^2 \left(2 - \tfrac{1}{8} r^4\right) r^5\, dr$$
$$= \left[\tfrac{1}{8}\left(\theta - \tfrac{1}{4}\sin 4\theta\right)\right]_0^{2\pi} \left[\tfrac{1}{3} r^6 - \tfrac{1}{80} r^{10}\right]_0^2$$
$$= \frac{\pi}{4} \left[\frac{1}{3}\cdot 2^6 - \frac{1}{5\cdot 2^4}\, 2^{10} \right]$$
$$= \frac{\pi}{4} \left(\frac{1}{3} - \frac{1}{5} \right)\cdot 2^6$$
$$= \frac{32}{15}\pi.$$





use cylindrical coord.

$0 \le \theta \le 2\pi$

$0 \le r \le 2$

[10] **7.** If $f(x,y)$ is continuous on $[a,b]\times[c,d]$ and

$$g(x,y)=\int_a^x\int_c^y f(s,t)\,dt\,ds$$

for $a<x<b,\ c<y<d$, show that $g_{xy}=g_{yx}=f(x,y)$.

Proof.

$$g_x=\frac{\partial}{\partial x}\left[\int_a^x\int_c^y f(s,t)\,dt\,ds\right]$$

$$=\frac{\partial}{\partial x}\left[\int_a^x\left(\int_c^y f(s,t)\,dt\right)ds\right]$$

$$=\int_c^y f(x,t)\,dt$$

$$g_{xy}=\frac{\partial g_x}{\partial y}=\frac{\partial}{\partial y}\left[\int_c^y f(x,t)\,dt\right]$$

$$=f(x,y)$$

$$g_y=\frac{\partial}{\partial y}\left[\int_a^x\int_c^y f(s,t)\,dt\,ds\right]$$

$$=\frac{\partial}{\partial y}\left[\int_c^y\left(\int_a^x f(s,t)\,ds\right)dt\right]$$

$$=\int_a^x f(s,y)\,ds$$

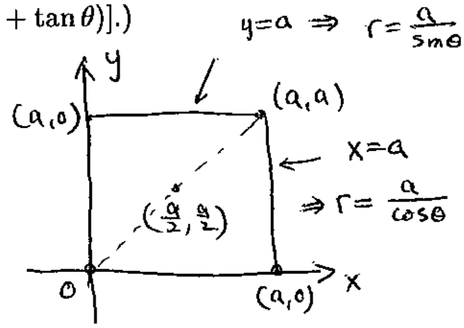
$$g_{yx}=\frac{\partial g_y}{\partial x}=\frac{\partial}{\partial x}\left[\int_a^x f(s,y)\,ds\right]$$

$$=f(x,y)$$

So. $g_{xy}=g_{yx}=f(x,y)$.

[10] **8.** The density at any point on a square lamina with edge length a is proportional to the distance from one corner of the square. If the density at the center of the square is $\frac{1}{a^2}$, find the mass of the lamina.

(**Note:** $\int \sec^3 d\theta = \frac{1}{2}[\,\sec\theta\tan\theta + \ln(\sec\theta + \tan\theta)].$)



Put one corner of the square lamina at the origin such that the two sides are on the x and y axes (see the figure). Then

$$\rho(x,y) = k\sqrt{x^2+y^2} = kr$$

$$\rho(\tfrac{a}{2},\tfrac{a}{2}) = \frac{1}{a^2} = k\sqrt{(\tfrac{a}{2})^2+(\tfrac{a}{2})^2} = k\frac{a}{\sqrt{2}} \Rightarrow k = \frac{\sqrt{2}}{a^3}.$$

$$m = \iint_D \rho(x,y)\,dA \qquad \text{(use polar coord.)}$$

$$= \int_0^{\pi/4}\int_0^{\frac{a}{\cos\theta}} kr^2\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{\frac{a}{\sin\theta}} kr^2\,dr\,d\theta$$

$$= k\int_0^{\frac{\pi}{4}} \frac{1}{3}\frac{a^3}{\cos^3\theta}\,d\theta + k\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{3}\frac{a^3}{\sin^3\theta}\,d\theta \quad \Downarrow\ \theta = \frac{\pi}{2}-\theta'$$

$$= \frac{ka^3}{3}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\theta}\,d\theta + \frac{ka^3}{3}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\theta'}\,d\theta'$$

$$= \frac{2ka^3}{3}\int_0^{\frac{\pi}{4}} \sec^3\theta\,d\theta$$

$$= \frac{2ka^3}{3}\cdot\frac{1}{2}\Big[\sec\theta\tan\theta + \ln(\sec\theta+\tan\theta)\Big]_0^{\frac{\pi}{4}}$$

$$= \frac{\cancel{2}\,\frac{\sqrt{2}}{\cancel{a^3}}\,\cancel{a^3}}{3}\,\frac{1}{\cancel{2}}\Big[\sqrt{2}+\ln(\sqrt{2}+1)\Big]$$

$$= \frac{\sqrt{2}}{3}\Big[\sqrt{2}+\ln(\sqrt{2}+1)\Big] \quad \text{(unit)}$$

[10] **BONUS** Question:

Find the surface area of the part of the cylinder $x^2 + y^2 = a^2$ between the plane $z = 0$ and the surface $z = x\,y$.

By symmetry, we only need to consider the part of the surface in the first octant, named S, and its projection on the yz-plane as D.

$$S: \quad x = \sqrt{a^2 - y^2}$$

$$D: \quad 0 \le z \le y\sqrt{a^2 - y^2}, \qquad 0 \le y \le a$$

$$\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} = \sqrt{1 + \left(\frac{-y}{2\sqrt{a^2 - y^2}}\right)^2 + 0} = \frac{a}{\sqrt{a^2 - y^2}}$$

$$A(S) = 4\iint_S dS = 4\iint_D \frac{a}{\sqrt{a^2 - y^2}}\,dy\,dz$$

$$= 4\int_0^a \frac{a}{\sqrt{a^2 - y^2}}\,dy \int_0^{y\sqrt{a^2-y^2}} dz$$

$$= 4\int_0^a \frac{a}{\cancel{\sqrt{a^2 - y^2}}} \cdot y\cancel{\sqrt{a^2 - y^2}}\,dy$$

$$= 4a\left[\frac{1}{2}y^2\right]_0^a$$

$$= 2a^3 \ (\text{unit}^2)$$