The Aggregate Production Function

These notes introduce the theory of the aggregate production function and present proofs for the three properties displayed by this function.

Definition*: *Factors of Production are the inputs to production from outside the business sector.

Factors of production can come from households, the government or the foreign sector. We will consider the household sector only. The principal factor inputs are:

1. labor services of different types,
2. capital services of different types, and
3. land services of different types.

An *intermediate good* is an input from another firm in the business sector. Although the theory of the aggregate production function can include intermediate goods, we develop the theory assuming there are no intermediate goods as the results are the same.

Definition: The aggregate production function is the maximum output that can be produced given the quantities of the factors of production.

Note that, in what follows, lower case letters refer to plant level variables while the corresponding capital letters refer to aggregate variables. Let output be denoted by Y, the capital service input by K, and the labor service input by N. We normalize so that K units of capital can provide K units of capital services. With this normalization we can then refer to both the capital stock and services of the capital stock by K. We normalize N in the analogous way. The aggregate production function F is then

$$Y = F(K, N).$$

This function displays the following three properties: the function

1. is increasing (possibly weakly),
2. displays constant returns to scale,
3. and displays diminishing returns.

We now develop the theory behind the aggregate production function and establish these three properties in Propositions 1, 2, and 3. There are plant (or production unit) technologies for producing output. Plant technologies differ in the amount of capital k and labor n that is used by that plant. The output of a type (k, n) plant is $y = f_{kn}$.

Any non-negative number of plants of a given type can be operated.

Assumption 1: For all possible plant types, k and n are infinitesimal with respect to the aggregate numbers K and N.

Assumption 2: All plant technologies require some strictly positive minimum amount of capital and/or labor.

***Assumption 3*:** There are a finite number of plant types or technologies.

Assumption 2 and 3 insure that there is a maximum output. Assumption 1 permits the number of plants of a given type to be a real number rather than an integer.

Definition: A production plan $z = \{z_{kn}\}$ is a vector specifying the number of plants of each type (k, n) that are operated.

Definition: A plan z is feasible given K and N if there are enough resources to operate the plan; thus a plan is feasible if

$$\sum_{k,n} k\, z_{kn} \le K \quad \text{and} \quad \sum_{k,n} n\, z_{kn} \le N .$$

Lemma 1: If plan z is feasible given K and N, then for any $\lambda > 0$ the plan λz is feasible given λK and λN. Here $\lambda z = \{\lambda z_{kn}\}$.

The output of plan z is

$$Y(z) = \sum_{k,n} f_{kn} z_{kn} .$$

The following lemma is an immediate result. Let $\lambda z = \{\lambda z_{kn}\}$.

Lemma 2: For $\lambda \ge 0$, $Y(\lambda z) = \lambda Y(z)$.

The value of the aggregate production function $F(K, N)$ is the maximum $Y(z)$ over all plans that are feasible given K and N, or

$$F(K, N) = \max_{z \geq 0} Y(z)$$

$$\text{subject to } \sum_{k,n} k\, z_{kn} \leq K \quad \text{and} \quad \sum_{k,n} n\, z_{kn} \leq N .$$

Proposition 1: **The aggregate production fu*nction F(K, N)* is (weakly) increasing.** Thus, if $K \geq K'$ and $N \geq N'$, then $F(K, N) \geq F(K', N')$.

Proof: Increasing the factor inputs increases the set of feasible plans. The maximum of a function over a bigger set is necessarily bigger (weakly). □

Definition: A function displays constant returns to scale if for any $\lambda > 0$, $f(\lambda x) = \lambda f(x)$.

Proposition 2: **Function *F(K,N)* displays constant returns to sc*ale.***

Proof. We first show

(1) $F(\lambda K, \lambda N) \geq \lambda F(K, N)$

The best plan given (K,N) can be scaled by factor λ and to produce $\lambda F(K, N)$ using $(\lambda K, \lambda N)$. The maximal feasible output given $(\lambda K, \lambda N)$ must be greater than or equal to all other feasible outputs given $(\lambda K, \lambda N)$ by definition.

We next show

(2) $F(K, N) \geq \lambda^{-1} F(\lambda K, \lambda N)$ or equivalently $\lambda F(K, N) \geq F(\lambda K, \lambda N)$.

The argument is the same as the previous one except that the roles of (K, N) and $(\lambda K, \lambda N)$ are interchanged. The best plan given $(\lambda K, \lambda N)$ can be scaled by factor λ^{-1} to produce $\lambda^{-1} F(\lambda K, \lambda N)$ using (K, N). The maximum feasible output given the resources must be greater than or equal to all other feasible those resources.

Inequalities (1) and (2) imply that $F(\lambda K, \lambda N) = \lambda F(K, N)$. □

Definition: A function is concave if $f(\frac{x_1+x_2}{2}) \geq f(\frac{x_1}{2}) + f(\frac{x_2}{2})$.

A function is concave if it lies above the line between any two points.

Proposition 3: **Function *F(K, N)* is concave.**

Proof. Let $X_1 = (K_1, N_1)$ and $X_2 = (K_2, N_2)$. First note

$$F(\frac{X_1+X_2}{2}) \leq Y(\frac{z_1+z_2}{2}) = \frac{1}{2}Y(z_1) + \frac{1}{2}Y(z_2) = \frac{1}{2}F(X_1) + \frac{1}{2}F(X_2)$$

Here z_1 and z_2 are the optimal plans for X_1 and X_2 respectively. The inequality follows from the fact $(z_1+z_2)/2$ is a feasible plan given $(X_1+X_2)/2$. The linearity of Y implies the first equality. The optimality of the z_i given the X_i imply the second equality. This establishes that the aggregate production function is concave.

Comment: The concavity of the aggregate production function implies that the marginal product of a factor is (weakly) decreasing.