Chapter 1 - Free Vibration of Multi-Degree-of-Freedom Systems - I

1.1 Free Undamped Vibration

The basic type of response of multi-degree-of-freedom systems is free undamped vibration. Analogous to single degree of freedom systems the analysis of free vibration yields the natural frequencies of the system. For the analysis, the elastic (restoring) properties of the system must be described first. This can either be done in terms of stiffness or flexibility

Structural Stiffness

Stiffness of a structure is described by the stiffness matrix, whose elements k_{ij} are defined as the force acting at node *i,* in order to produce a sole unit displacement at node *j.* In "lumped mass" models, the stiffness constants defined above are identical to the stiffness used in static models

Example - Multi-storey "shear" building

A shear building is one where the resistance to lateral loads is from the bending of the columns – the floors are infinitely rigid – and the columns are fixed-ended where connected to the floors.

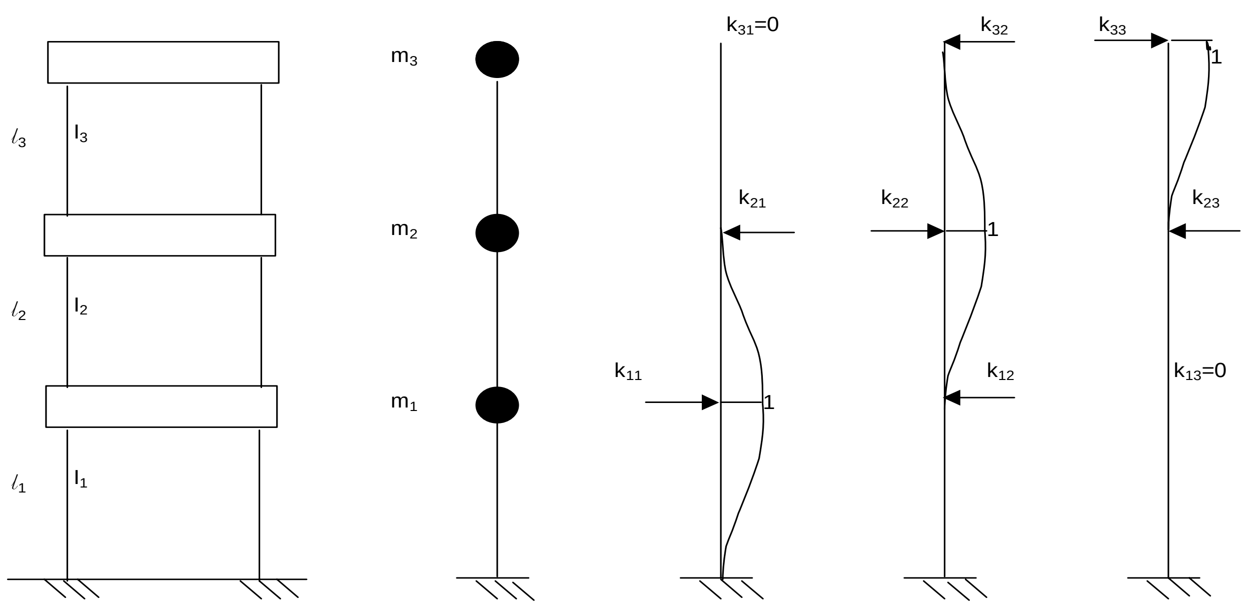


Fig. 1.1 Multi Storey Shear Building

The basic stiffness constant for a column subjected to shear only is $k = 12\dfrac{EI}{\ell^3}$, where *EI* is the bending stiffness of the column and ℓ is the column length. The assembly of the stiffness matrix is performed one element at a time, with each floor of the building sequentially subjected to a unit shear displacement and the stiffnesses added as appropriate. e.g. the stiffness element for the first floor of the shear building in Fig. 1.1, due to a unit displacement of the first floor is:

$$k_{11} = 2 * \frac{12EI_1}{\ell_1^3} + 2 * \frac{12EI_2}{\ell_2^3}$$

(the '2' multiplier is for 2 columns per storey)

The full stiffness matrix for the 3-storey shear building is:

$$\begin{vmatrix} k_{11} = 2 * \dfrac{12EI_1}{\ell_1^3} + 2 * \dfrac{12EI_2}{\ell_2^3} & k_{12} = 2 * \left(-\dfrac{12EI_2}{\ell_2^3} \right) & k_{13} = 0 \\ k_{21} = 2 * \left(-\dfrac{12EI_2}{\ell_2^3} \right) & k_{22} = 2 * \dfrac{12EI_2}{\ell_2^3} + 2 * \dfrac{12EI_3}{\ell_3^3} & k_{23} = 2 * \left(-\dfrac{12EI_3}{\ell_3^3} \right) \\ k_{31} = 0 & k_{32} = 2 * \left(-\dfrac{12EI_3}{\ell_3^3} \right) & k_{33} = 2 * \dfrac{12EI_3}{\ell_3^3} \end{vmatrix}$$

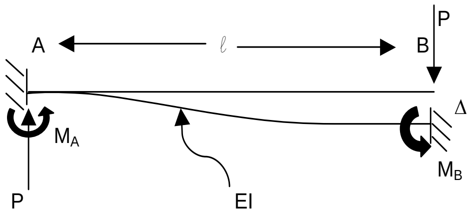
Note: - the matrix is diagonally symmetric ($k_{12} = k_{21}$)

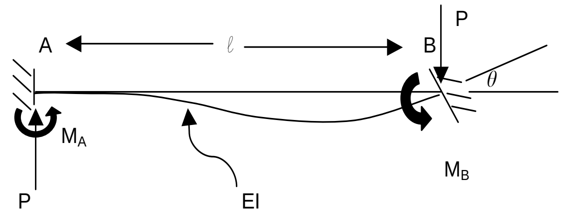
<u>Static Condensation</u>

This is the term given to the simplification of a stiffness matrix through the elimination of degrees of freedom. For example, in most buildings and structures exposed to lateral loads, there are no significant external moments or mass moment of inertia acting in the joints. Therefore the joint rotations can be eliminated from the governing equations, so the deformation of the structure can be expressed in terms of lateral displacements only.

Considering the full 4x4 stiffness matrix for the column shown below, the elements can be assembled one degree of freedom at a time. We shall see how this can be simplified using *Static Condensation.*

Recall the stiffness characteristics of a fixed ended beam:





$$M_A = M_B = 6EI\Delta/\ell^2$$
$$P\ell = M_A + M_B = 2*6EI\Delta/\ell^2$$
$$P = 12EI\Delta/\ell^3$$

$$\boxed{\begin{array}{l}\dfrac{P}{\Delta} = \dfrac{12EI}{\ell^3}\\[2mm] \dfrac{M}{\Delta} = \dfrac{6EI}{\ell^2}\end{array}}$$

$$M_A = \tfrac{1}{2}M_B = \frac{2EI\theta}{\ell}$$
$$P\ell = M_B + M_A = \frac{6EI\theta}{\ell}$$
$$P = \frac{6EI\theta}{\ell^2}$$

$$\boxed{\begin{array}{l}\dfrac{P}{\theta} = \dfrac{6EI}{\ell^2}\\[2mm] \dfrac{M_B}{\theta} = \dfrac{4EI}{\ell}\\[2mm] \dfrac{M_A}{\theta} = \dfrac{2EI}{\ell}\end{array}}$$

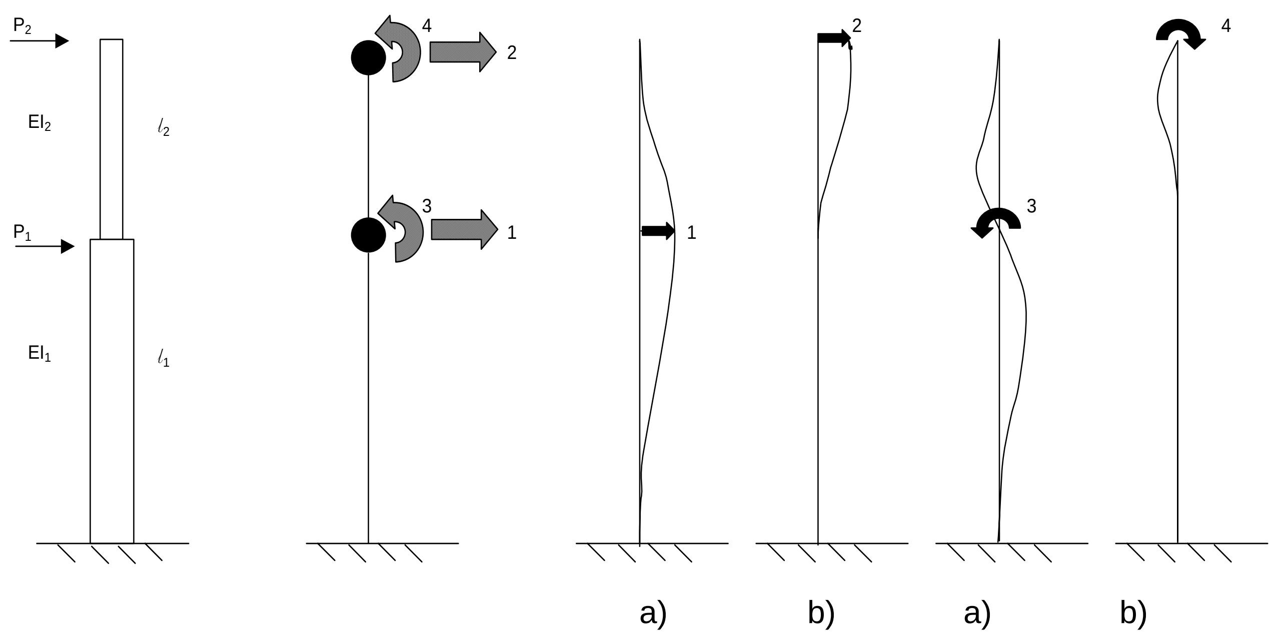


Fig. 5.2 a) Generation of full stiffness matrix (4x4) and b) Condensed (2x2)

Assembling the elements of the complete stiffness matrix, we obtain:

$$k_{11} = \frac{12EI_1}{\ell_1^{\,3}} + \frac{12EI_2}{\ell_2^{\,3}}$$

$$k_{12} = k_{21} = -\frac{12EI_2}{\ell_2^{\,3}}$$

$$k_{13} = k_{31} = \frac{6EI_1}{\ell_1^2} - \frac{6EI_2}{\ell_2^2}$$

$$k_{14} = k_{41} = -\frac{6EI_2}{\ell_2^2}$$

$$k_{22} = \frac{12EI_2}{\ell_2^3}$$

$$k_{23} = k_{32} = \frac{6EI_2}{\ell_2^2}$$

$$k_{24} = k_{42} = \frac{6EI_2}{\ell_2^2}$$

$$k_{33} = \frac{4EI_1}{\ell_1} + \frac{4EI_2}{\ell_2}$$

$$k_{34} = k_{43} = \frac{2EI_2}{\ell_2}$$

$$k_{44} = \frac{4EI_2}{\ell_2}$$

Now, if only static horizontal forces $P_{1,2}$ act, the matrix equation relating the input forces to the output displacements and rotations is:

$$\begin{Bmatrix} P_1 \\ P_2 \\ \hline 0 \\ 0 \end{Bmatrix} = \left[\begin{array}{cc|cc} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ \hline k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{array}\right] \begin{Bmatrix} u_1 \\ u_2 \\ \hline \psi_1 \\ \psi_2 \end{Bmatrix}$$

or, generally:

$$\begin{Bmatrix} P \\ \hline 0 \end{Bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right] \begin{Bmatrix} u \\ \hline \psi \end{Bmatrix}$$

note that the lower part of the equation:

$$\{0\} = [B]^T \{u\} + [C]\{\psi\}$$

yields the rotations:

$$\{\psi\} = -[C]^{-1}[B]^T\{u\}$$

then substituting this expression for the unknown rotations in terms of the unknown displacements into the upper part of the equation:

$$\{P\} = [A]\{u\} - [B][C]^{-1}[B]^T\{u\}$$
$$\{P\} = \left([A] - [B][C]^{-1}[B]^T\right)\{u\}$$

the *Condensed Stiffness Matrix* is then:

$$[k'] = \left([A] - [B][C]^{-1}[B]^T\right)$$

In this case, it is a n x n matrix, where n is the number of translational degrees of freedom (in this case a 2 x 2 matrix) involving translations only. If the rotations are desired, it is a simple matter to insert the resulting displacements into the known relationship between displacement and rotation.

<u>Governing Equations for the Solution to the Free Vibration Problem in n Degrees-of-Freedom</u>

With the stiffness constants defined, the governing equations of motion can be written using Newton's Second Law for each of the masses in the system:

$$\text{i.e. mass} * \text{acceleration} = \sum \text{forces acting on the mass}$$

$$m_1\ddot{u}_1 = -k_{11}u_1 - k_{12}u_2 \ldots - k_{1i}u_i \ldots - k_{1n}u_n$$
$$m_2\ddot{u}_2 = -k_{21}u_1 - k_{22}u_2 \ldots - k_{2i}u_i \ldots - k_{2n}u_n$$
.
.
$$m_n\ddot{u}_n = -k_{n1}u_1 - k_{n2}u_2 \ldots - k_{ni}u_i \ldots - k_{nn}u_n$$

where:

$$\ddot{u}_i = \frac{d^2u_i(t)}{dt^2}$$

Then for mass m_i , in general for j = 1,2…n where n is the number of the degrees of freedom:

$$m_i \ddot{u}_i + \sum_{j=1}^{n} k_{ij} u_j = 0 \qquad (1.1)$$

This is a set of simultaneous, ordinary differential equations of the second order. This can be written in matrix form:

$$[m]\{\ddot{u}\} + [k]\{u\} = \{0\} \qquad (1.2)$$

where $[m]$ is the diagonal mass matrix: $[m] = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 \\ 0 & 0 & m_i & 0 \\ 0 & 0 & 0 & m_n \end{bmatrix}$

The displacement vector is a column matrix:

$$\{u(t)\} = \begin{Bmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ u_n(t) \end{Bmatrix} = [u_1(t) \quad u_2(t) \quad . \quad . \quad u_n(t)]^T$$

and the symmetrical stiffness matrix [k] is:

$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{1j} & k_{1n} \\ k_{21} & k_{22} & k_{2j} & k_{2n} \\ k_{j1} & k_{j2} & k_{jj} & k_{jn} \\ k_{n1} & k_{n2} & k_{nj} & k_{nn} \end{bmatrix}$$

As in the single degree of freedom case, the *particular solution* is:

$$u_i(t) = u_i \sin \omega t$$
$$\ddot{u}_i(t) = -u_i \omega^2 \sin \omega t$$

or, in matrix notation:

$$\{u(t)\} = \{u\} \sin \omega t$$
$$\{\ddot{u}(t)\} = -\omega^2 \{u\} \sin \omega t \qquad (1.3)$$

In which $\{u\}$ is the column vector of amplitudes, which are independent of time. Subsituting equation (1.3) into equation (1.2), yields:

$$-[m]\omega^2\{\ddot{u}\}\sin\omega t + [k]\{u\}\sin\omega t = \{0\}$$

The column vector of amplitudes is:

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_n \end{Bmatrix} = [u_1 \quad u_2 \quad . \quad . \quad u_n]^T \qquad (1.4)$$

This is a homogeneous algebraic equation for u_i, where the frequency, ω is unknown. This is called the "*Eigenvalue Problem"* The solution to the Eigenvalue Problem utilizes the basic properties of homogeneous algebraic equations which imply that the roots (unknowns) are nontrivial (i.e. not equal to zero) only if the determinant of the coefficients vanishes.

$$([k] - \omega^2[m])\{u\} = \{0\}$$

The roots of this equation are non-zero <u>only</u> if the determinant is zero. i.e.:

$$|\,[k] - \omega^2[m]\,| = 0$$

if $\lambda = 1/\omega^2$ and we pre-multiply by $-\lambda[k]^{-1}$, then:

$$(-\lambda[k]^{-1}[k] + [k]^{-1}[m])\{u\} = \{0\}$$
$$(-\lambda[\mathrm{I}] + [k]^{-1}[m])\{u\} = \{0\}$$

where $[\mathrm{I}]$ is the Identity Matrix. A non-trivial solution only exists if the determinant is equal to zero:

$$|-\lambda[\mathrm{I}] + [k]^{-1}[m]| = 0$$

To find the values of λ which satisfy this equation results in a set of unique *Eigenvalues* or Natural Frequencies. After the Eigenvalues have been determined, they are substituted back into the Homogeneous Equation involving $\{u\}$. For each value of λ (recall that $\lambda = 1/\omega^2$) or natural frequency, a complete set of dimensionless displacements are obtained, one for each degree of freedom. There are the mode shapes associated with each mode of vibration, called *Eigenvectors*.

In our two-degree of freedom "flagpole" problem, the solution of these equations results in a closed-form solution for ω^2, as follows:

Recall that we now have a 2x2 Condensed Stiffness Matrix, $[k']$, and the equation for the characteristic determinant:

$$\left| [k'] - \omega^2 [m] \right| = 0$$

becomes:

$$\begin{vmatrix} k'_{11} - m_1\omega^2 & k'_{12} \\ k'_{21} & k'_{22} - m_2\omega^2 \end{vmatrix} = 0$$

the determinant is then a quadratic equation in ω^2:

$$\Delta = (k'_{11} - m_1\omega^2)(k'_{22} - m_2\omega^2) - k'_{12}k'_{21} = 0$$
$$\Delta = \omega^4 m_1 m_2 - \omega^2(m_1 k'_{22} + m_2 k'_{11}) + k'_{11}k'_{22} - k'_{12}k'_{21} = 0$$
$$\omega^4 - \left(\frac{k'_{11}}{m_1} + \frac{k'_{22}}{m_2}\right)\omega^2 - \frac{k'_{12}}{m_1 m_2} = 0$$

the solution of which is:

$$\omega_{1,2}{}^2 = \frac{1}{2}\left(\frac{k'_{11}}{m_1} + \frac{k'_{22}}{m_2}\right) \pm \sqrt{\frac{1}{4}\left(\frac{k'_{11}}{m_1} + \frac{k'_{22}}{m_2}\right) + \frac{k'_{12}}{m_1 m_2}}$$

With each frequency, the amplitude ratios, or mode shapes can be calculated from:

$$\left([k'] - \omega^2[m]\right)\{u\} = \{0\}$$

$$\left(\begin{bmatrix} k'_{11} & k'_{12} \\ k'_{21} & k'_{22} \end{bmatrix} - \omega^2{}_j \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} k'_{11} - \omega^2{}_j m_1 & k'_{12} \\ k'_{21} & k'_{22} - \omega^2{}_j m_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

This results in two equations and two unknowns:

$$\left(k'_{11} - \omega_j^2 m_1\right)u_1 + k'_{12}u_2 = 0$$
$$k'_{21}u_1 + \left(k'_{22} - \omega_j^2 m_2\right)u_2 = 0$$

if we define the relative displacement as $a_j = \frac{u_2}{u_1}$, then

$$a_j = \frac{u_2}{u_1} = \frac{\left(\omega_j^2 m_1 - k'_{11}\right)}{k_{12}} = \frac{k_{12}}{\left(\omega_j^2 m_2 - k'_{22}\right)} \quad \text{and} \quad a_1 a_2 = \frac{m_1}{m_2}$$

Both of these equations must yield the same answer, which acts as a check on the resulting mode shapes. With more degrees-of-freedom than two, it is desirable to use a computer to solve the problem.